Classification of polarized manifolds by the second sectional Betti number, II *†‡

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Abstract

Let X be an n-dimensional smooth projective variety defined over the field of complex numbers, let L be a very ample line bundle on X. Then we classify (X, L) with $b_2(X, L) = h^2(X, \mathbb{C}) + 2$, where $b_2(X, L)$ is the second sectional Betti number of (X, L).

1 Introduction

Let X be a smooth projective variety of dimension n defined over the field of complex numbers $\mathbb{C}$ and let L be an ample line bundle on X. Then we call this pair (X, L) a polarized manifold. In [11], for every integer i with $0 \leq i \leq n$, we defined the invariant $b_i(X, L)$ which is called the *ith sectional Betti number of* (X, L). If L is spanned, then we can prove that $b_i(X, L) \geq h^i(X, \mathbb{C})$ (see Remark 2.1.1 (iii.1) below). So it is interesting to classify (X, L) by the value of $b_i(X, L) - h^i(X, \mathbb{C})$.

In this paper, we consider the case of $i = 2$. Then in [12, Theorem 4.1] (resp. [14, Theorem 3.1]) we have classified polarized manifolds (X, L) such that L is spanned and $b_2(X, L) = h^2(X, \mathbb{C})$ (resp. $b_2(X, L) = h^2(X, \mathbb{C}) + 1$).

So in this paper, as the next step, we will classify polarized manifolds (X, L) such that L is very ample and $b_2(X, L) = h^2(X, \mathbb{C}) + 2$.

In this paper we will use the customary notation in algebraic geometry.

2 Preliminaries

2.1 Review on sectional invariants of polarized manifolds

In this subsection, we will review the theory of sectional invariants of polarized manifolds which will be used in the main theorem (Theorem 3.1) and its proof.

Notation 2.1.1 (1) Let X be a projective variety of dimension n, let L be an ample line bundle on X. Then the Euler-Poincaré characteristic $\chi(L^{\otimes t})$ of $L^{\otimes t}$ is a polynomial in t of degree n, and we can describe $\chi(L^{\otimes t})$ as follows.

$$\chi(L^{\otimes t}) = \sum_{j=0}^{n} \chi_j(X, L)\binom{t+j-1}{j}.$$

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(2) Let Y be a smooth projective variety of dimension i, let $\mathcal{T}_Y$ be the tangent bundle of Y, and let Ω_Y be the dual bundle of $\mathcal{T}_Y$. For every integer j with $0 \leq j \leq i$, we put

$$\begin{aligned} h_{i,j}(c_1(Y), \cdots, c_i(Y)) &:= \chi(\Omega_Y^j) \\ &= \int_Y \mathrm{ch}(\Omega_Y^j)\mathrm{Td}(\mathcal{T}_Y). \end{aligned}$$

(Here $\mathrm{ch}(\Omega_Y^j)$ (resp. $\mathrm{Td}(\mathcal{T}_Y)$) denotes the Chern character of Ω_Y^j (resp. the Todd class of $\mathcal{T}_Y$). See [15, Examples 3.2.3 and 3.2.4].)

(3) Let (X, L) be a polarized manifold of dimension n. For every integers i and j with $0 \leq j \leq i \leq n$, we put

$$C_j^i(X, L) := \sum_{l=0}^{j} (-1)^l \binom{n-i+l-1}{l} c_{j-l}(X) L^l,$$

$$w_i^j(X, L) := h_{i,j}(C_1^i(X, L), \cdots, C_i^i(X, L)) L^{n-i}.$$

(4) Let X be a smooth projective variety of dimension n. For every integers i and j with $0 \leq j \leq i \leq n$, we put

$$H_1(i, j) := \begin{cases} \sum_{s=0}^{i-j-1} (-1)^s h^s(\Omega_X^j) & \text{if } j \neq i, \\ 0 & \text{if } j = i, \end{cases}$$

$$H_2(i, j) := \begin{cases} \sum_{t=0}^{j-1} (-1)^{i-t} h^t(\Omega_X^{i-j}) & \text{if } j \neq 0, \\ 0 & \text{if } j = 0. \end{cases}$$

Definition 2.1.1 (See [10, Definition 2.1] and [11, Definition 3.1].) Let (X, L) be a polarized manifold of dimension n, and let i and j be integers with $0 \leq j \leq i \leq n$. (Here we use Notation 2.1.1.)

(1) The *ith sectional geometric genus* $g_i(X, L)$ *of* (X, L) is defined as follows:

$$g_i(X, L) := (-1)^i (\chi_{n-i}(X, L) - \chi(\mathcal{O}_X)) + \sum_{j=0}^{n-i} (-1)^{n-i-j} h^{n-j}(\mathcal{O}_X).$$

(2) The *ith sectional Euler number* $e_i(X, L)$ *of* (X, L) is defined by the following:

$$e_i(X, L) := C_i^i(X, L) L^{n-i}.$$

(3) The *ith sectional Betti number* $b_i(X, L)$ *of* (X, L) is defined by the following:

$$b_i(X, L) := \begin{cases} e_0(X, L) & \text{if } i = 0, \\ (-1)^i \left(e_i(X, L) - \sum_{j=0}^{i-1} 2(-1)^j h^j(X, \mathbb{C}) \right) & \text{if } 1 \leq i \leq n. \end{cases}$$

(4) The *ith sectional Hodge number* $h_i^{j,i-j}(X, L)$ *of type* $(j, i-j)$ *of* (X, L) is defined by the following:

$$h_i^{j,i-j}(X, L) := (-1)^{i-j} \left\{ w_i^j(X, L) - H_1(i, j) - H_2(i, j) \right\}.$$

Remark 2.1.1 (i) For every integers i and j with $0 \leq j \leq i \leq n$, $g_i(X, L)$, $e_i(X, L)$, $b_i(X, L)$ and $h_i^{j,i-j}(X, L)$ are integer (see [11, Proposition 3.1]).

(ii) Let (X, L) be a polarized manifold of dimension n. For every integers i and j with $0 \leq j \leq i \leq n$, we get the following (see [11, Theorem 3.1]).

(ii.1) $b_i(X, L) = \sum_{k=0}^{i} h_i^{k,i-k}(X, L)$.

(ii.2) $h_i^{j,i-j}(X, L) = h_i^{i-j,j}(X, L)$.

(ii.3) $h_i^{i,0}(X, L) = h_i^{0,i}(X, L) = g_i(X, L)$.

(iii) Assume that L is ample and spanned. Then, for every integers i and j with $0 \leq j \leq i \leq n$, the following inequalities hold (see [10, Theorem 3.1] and [11, Proposition 3.3]).

(iii.1) $b_i(X, L) \geq h^i(X, \mathbb{C})$.

(iii.2) $h_i^{j,i-j}(X, L) \geq h^{j,i-j}(X)$.

(iii.3) $g_i(X, L) \geq h^i(\mathcal{O}_X)$.

2.2 Adjunction theory of polarized manifolds

In this subsection, we will review the adjunction theory which will be used later.

Definition 2.2.1 Let (X, L) be a polarized manifold of dimension n.

(1) We say that (X, L) is a *scroll* (resp. *quadric fibration*, *Del Pezzo fibration*) *over a normal projective variety* Y *of dimension* m with $1 \leq m < n$ (resp. $1 \leq m < n$, $1 \leq m < n-1$) if there exists a surjective morphism with connected fibers $f : X \to Y$ such that $K_X + (n-m+1)L = f^*A$ (resp. $K_X + (n-m)L = f^*A$, $K_X + (n-m-1)L = f^*A$) for some ample line bundle A on Y.

(2) (X, L) is called a *classical scroll over a normal variety* Y if there exists a vector bundle $\mathcal{E}$ on Y such that $X \cong \mathbb{P}_Y(\mathcal{E})$ and $L = H(\mathcal{E})$, where $H(\mathcal{E})$ is the tautological line bundle.

(3) We say that (X, L) is a *hyperquadric fibration over a smooth projective curve* C if (X, L) is a quadric fibration over C such that the morphism $f : X \to C$ is the contraction morphism of an extremal ray. In this case, $(F, L_F) \cong (\mathbb{Q}^{n-1}, \mathcal{O}_{\mathbb{Q}^{n-1}}(1))$ for any general fiber F of f, every fiber of f is irreducible and reduced (see [18] or [7, Claim (3.1)]) and $h^2(X, \mathbb{C}) = 2$.

Remark 2.2.1 (1) If (X, L) is a scroll over a smooth projective curve C, then (X, L) is a classical scroll over C (see [1, Proposition 3.2.1]).

(2) If (X, L) is a scroll over a normal projective surface S, then S is smooth and (X, L) is also a classical scroll over S (see [3, (3.2.1) Theorem] and [9, (11.8.6)]).

(3) Assume that (X, L) is a quadric fibration over a smooth curve C with $\dim X = n \geq 3$. Let $f : X \to C$ be its morphism. By [3, (3.2.6) Theorem] and the proof of [18, Lemma (c) in Section 1], we see that (X, L) is one of the following:

(a) A hyperquadric fibration over C.

(b) A classical scroll over a smooth surface with $\dim X = 3$.

Theorem 2.2.1 *Let* (X, L) *be a polarized manifold with* $\dim X = n \geq 3$. *Then* (X, L) *is one of the following types.*

(1) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$.

(2) $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1))$.

(3) *A scroll over a smooth projective curve.*

(4) $K_X \sim -(n-1)L$*, that is,* (X, L) *is a Del Pezzo manifold.*

(5) *A hyperquadric fibration over a smooth projective curve.*

(6) *A classical scroll over a smooth projective surface.*

(7) *Let* (M, A) *be a reduction of* (X, L)*.*

   (7.1) $n = 4$, $(M, A) = (\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2))$.

   (7.2) $n = 3$, $(M, A) = (\mathbb{Q}^3, \mathcal{O}_{\mathbb{Q}^3}(2))$.

   (7.3) $n = 3$, $(M, A) = (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3))$.

   (7.4) $n = 3$*,* M *is a* $\mathbb{P}^2$*-bundle over a smooth curve* C*, the nef value of* A *is* $\frac{3}{2}$*, and* $(F', A|_{F'}) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ *for any fiber* F' *of it.*

   (7.5) $K_M + (n-2)A$ *is nef.*

Proof. See [1, Proposition 7.2.2, Theorems 7.2.4, 7.3.2 and 7.3.4] and [9, (11.2), (11.7), and (11.8)]. □

Notation 2.2.1 (1) Let (X, L) be a hyperquadric fibration over a smooth curve C and let $f : X \to C$ be its morphism. We put $\mathcal{E} := f_*(L)$. Then $\mathcal{E}$ is a locally free sheaf of rank $n+1$ on C. Let $\pi : \mathbb{P}_C(\mathcal{E}) \to C$ be the projective bundle. Then $X \in |2H(\mathcal{E}) + \pi^*(B)|$ for some $B \in \mathrm{Pic}(C)$ and $L = H(\mathcal{E})|_X$, where $H(\mathcal{E})$ is the tautological line bundle of $\mathbb{P}_C(\mathcal{E})$. We put $e := \deg \mathcal{E}$ and $b := \deg B$.

(2) (See [9, (13.10)].) Let (M, A) be a $\mathbb{P}^2$-bundle over a smooth curve C and $A|_F = \mathcal{O}_{\mathbb{P}^2}(2)$ for any fiber F of it. Let $f : M \to C$ be the fibration and $\mathcal{E} := f_*(K_M + 2A)$. Then $\mathcal{E}$ is a locally free sheaf of rank 3 on C, and $M \cong \mathbb{P}_C(\mathcal{E})$ such that $H(\mathcal{E}) = K_M + 2A$. In this case, $A = 2H(\mathcal{E}) + f^*(B)$ for a line bundle B on C, and by the canonical bundle formula we have $K_M = -3H(\mathcal{E}) + f^*(K_C + \det\mathcal{E})$. Here we set $e := \deg \mathcal{E}$ and $b := \deg B$.

2.3 A classification of very ample vector bundles $\mathcal{E}$ on surfaces with $c_2(\mathcal{E}) = 3$

Here we classify very ample vector bundles $\mathcal{E}$ on smooth projective surfaces with $c_2(\mathcal{E}) = 3$. We will use this result later.

Theorem 2.3.1 *Let* S *be a smooth projective surface and let* $\mathcal{E}$ *be a very ample vector bundle on* S *with* $c_2(\mathcal{E}) = 3$ *and* $\mathrm{rank}\mathcal{E} \geq 2$*. Then* $(S, \mathcal{E})$ *is one of the following types.*

(i) $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 3})$.

(ii) $(\mathbb{P}^2, T_{\mathbb{P}^2})$*, where* $T_{\mathbb{P}^2}$ *is the tangent bundle of* $\mathbb{P}^2$*.*

(iii) $(\mathbb{P}^1 \times \mathbb{P}^1, [p_1^*(\mathcal{O}_{\mathbb{P}^1}(1)) \otimes p_2^*(\mathcal{O}_{\mathbb{P}^1}(2))] \oplus [p_1^*(\mathcal{O}_{\mathbb{P}^1}(1)) \otimes p_2^*(\mathcal{O}_{\mathbb{P}^1}(1))])$*, where* p_i *is the* i*th projection.*

(iv) S *is a blowing up of* $\mathbb{P}^2$ *at a point and* $\mathcal{E} = (p^*(\mathcal{O}_{\mathbb{P}^2}(2)) - E)^{\oplus 2}$*, where* $p : S \to \mathbb{P}^2$ *is the morphism and* E *is the exceptional divisor of* p*.*

(v) $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(3))$.

(vi) *S is a Del Pezzo surface of degree* 3 *and* $\mathcal{E} \cong \mathcal{O}(-K_S)^{\oplus 2}$.

Proof. By a result of Noma [22, Corollary], we see that $(S, \mathcal{E}) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(3))$ if $c_1(\mathcal{E})^2 \geq 4c_2(\mathcal{E})+1 = 13$. So we may assume that $c_1(\mathcal{E})^2 \leq 12$. We consider $(\mathbb{P}_S(\mathcal{E}), H(\mathcal{E}))$ and let $X := \mathbb{P}_S(\mathcal{E})$, $L := H(\mathcal{E})$ and $n := \dim X$. Then $H(\mathcal{E})$ is very ample and $H(\mathcal{E})^n = c_1(\mathcal{E})^2 - c_2(\mathcal{E}) \leq 12-3 = 9$. Let $\pi : \mathbb{P}_S(\mathcal{E}) \to S$ be the projection. We use a classification of polarized manifolds by the degree (see [17], [19] and [5]). First of all, we prove the following claim.

Claim 2.3.1 *If $g(X, L) \leq 3$ and $c_2(\mathcal{E}) = 3$, then $(S, \mathcal{E})$ is one of the types* (i), (ii), (iii), (iv) *and* (v) *in Theorem* 2.3.1.

Proof. Using [8], [4] and [20, Corollary 4.7], we get the assertion. □

From now on, we assume that $g(X, L) \geq 4$. By the list of [17], we have $L^n \geq 6$.

(A) The case where $L^n = 6$. Then we see from the list of [17] that X is either a complete inetersection of type $(2, 3)$ or a hypersurface in $\mathbb{P}^{n+1}$. But in each case we have $\mathrm{Pic}(X) \cong \mathbb{Z}$ and this is impossible.

(B) The case where $L^n = 7$. Then we see from the list of [17] and Table II of [2, Page 55] that (X, L) is one of the following types.

(B.1) $(X, L) = (\mathbb{P}_T(\mathcal{F}), H(\mathcal{F}))$ and $g(X, L) = 4$, where T is the blowing up of $\mathbb{P}^2$ at 6 points and $\mathcal{F}$ is a locally free sheaf on T.

(B.2) $n = 3$, $g(X, L) = 5$ and $\sigma_P : X \to Y$ is the blowing up of Y at a point P, where Y is a smooth complete intersection of type $(2, 2, 2)$.

(B.3) $g(X, L) = 6$ and the morphism $\phi : X \to \mathbb{P}^1$ defined by the complete linear system $|K_X + L|$ is a fibration over $\mathbb{P}^1$.

(B.4) X is a hypersurface of degree 7 in $\mathbb{P}^{n+1}$.

(B.I) First we consider the case (B.2). Then $\mathrm{Pic}(X) \cong \mathbb{Z}^{\oplus 2}$ and $\mathrm{Pic}(S) \cong \mathbb{Z}$. Next we prove the following.

Claim 2.3.2 $\kappa(S) = -\infty$ *holds.*

Proof. In this case, there exists an effective divisor E on X such that $E \cong \mathbb{P}^2$. We note that $\pi(E)$ is not a point because every fiber of π is $\mathbb{P}^1$. Therefore $\pi_E : E \to S$ is surjective because $E \cong \mathbb{P}^2$. Assume that π_E is not finite. Then there exists a fiber F_π of π such that F_π is contracted by σ_P. Hence [1, Lemma 4.1.13] there exists a morphism $\delta : S \to Y$ such that $\sigma_P = \delta \circ \pi$. But this is impossible because σ_P is surjective and $\dim S < \dim Y$. Therefore π_E is finite and we have $\kappa(S) = -\infty$ because $\kappa(E) = -\infty$. □

We see from Claim 2.3.2 and $\mathrm{Pic}(S) \cong \mathbb{Z}$ that $S \cong \mathbb{P}^2$. We note that $\mathrm{rank}\mathcal{E} = 2$ because $\dim X = 3$ in this case. Hence by [20, Corollary 4.7] $g(X, L) \leq 3$ holds and this case is ruled out.
(B.II) Next we consider the case (B.3). Since $h^0(K_X + L) = h^0(K_{\mathbb{P}_S(\mathcal{E})} + H(\mathcal{E})) = 0$, this case is also ruled out.
(B.III) Next we consider the case (B.4). This case is also ruled out because $\mathrm{Pic}(X) \not\cong \mathbb{Z}$.
(B.IV) Finally we consider the case (B.1). Then we have $\mathrm{Pic}(T) \cong \mathbb{Z}^{\oplus 7}$, $\mathrm{Pic}(X) \cong \mathbb{Z}^{\oplus 8}$ and $\mathrm{Pic}(S) \cong \mathbb{Z}^{\oplus 7}$. Since $c_2(\mathcal{E}) = 3$ and $L^n = 7$, we have $c_1(\mathcal{E})^2 = 10$. Hence we have $K_S c_1(\mathcal{E}) = -4$ because $g(S, c_1(\mathcal{E})) = g(X, L) = 4$. Next we prove the following.

Claim 2.3.3 $\kappa(S) = -\infty$ *holds.*

Proof. Let $\rho : X = \mathbb{P}_T(\mathcal{F}) \to T$ be the projection. Let $D_1, \dots, D_{n-2}$ be general members of $|L|$ such that $X_{n-2} := D_1 \cap \dots \cap D_{n-2}$ is a smooth projective surface. Here we note that $\rho_{X_{n-2}} : X_{n-2} \to T$ and $\pi_{X_{n-2}} : X_{n-2} \to S$ are birational because $L^{n-2}F_\rho = 1$ (resp. $L^{n-2}F_\pi = 1$) for any general fiber F_ρ (resp. F_π) of ρ (resp. π). Therefore S is birationally equivalent to T. So we get the assertion because $\kappa(T) = -\infty$. □

Since $\kappa(S) = -\infty$, $h^1(\mathcal{O}_S) = 0$ and $\mathrm{Pic}(S) \cong \mathbb{Z}^{\oplus 7}$, we see that $K_S^2 = 3$. Hence we get

$$(K_S c_1(\mathcal{E}))^2 = 16 < 30 = (K_S)^2 (c_1(\mathcal{E}))^2,$$

but this contradicts the Hodge index theorem. Therefore this case is also impossible.

(C) The case where $L^n = 8$. Then since we assume that $g(X, L) \geq 4$, we see from the list of [19] that (X, L) is one of the following types.

(C.1) $(X, L) = (\mathbb{P}_{\mathbb{Q}^2}(\mathcal{F}), H(\mathcal{F}))$ and $g(X, L) = 4$, where $\mathcal{F}$ is a locally free sheaf of rank two on $\mathbb{Q}^2$.

(C.2) X is a smooth complete intersection of type $(2, 2, 2)$.

(C.3) The morphism $\phi : X \to \mathbb{P}^1$ defined by $|K_X + L|$ is a fibration over $\mathbb{P}^1$.

(C.4) X is a complete intersection of type $(2, 4)$.

(C.5) X is a hypersurface of degree 8 in $\mathbb{P}^{n+1}$.

(C.I) First we consider the cases (C.2), (C.4) and (C.5). These cases are ruled out because $\mathrm{Pic}X \ncong \mathbb{Z}$.
(C.II) Next we consider the case (C.3). Since $h^0(K_X + L) = h^0(K_{\mathbb{P}_S(\mathcal{E})} + H(\mathcal{E})) = 0$, this case is also ruled out.
(C.III) Finally we consider the case (C.1). Since $g(S, c_1(\mathcal{E})) = g(X, L) = 4$ and $c_1(\mathcal{E})^2 = 11$, we have $K_S c_1(\mathcal{E}) = -5$. Moreover $\mathrm{Pic}(X) \cong \mathbb{Z}^{\oplus 3}$ and $h^1(\mathcal{O}_X) = 0$. Hence we have $\mathrm{Pic}(S) \cong \mathbb{Z}^{\oplus 2}$ and $h^1(\mathcal{O}_S) = 0$. By the same argument as in the proof of Claim 2.3.3, we see that $\kappa(S) = -\infty$. So we have $K_S^2 = 8$, and

$$(K_S c_1(\mathcal{E}))^2 = 25 < 88 = (K_S)^2 (c_1(\mathcal{E}))^2.$$

But this contradicts the Hodge index theorem. Therefore this case is also impossible.

(D) The case where $L^n = 9$. In this case, since we assume that $g(X, L) \geq 4$, we see from [6, Table III in page 104] (see also [5]) that (X, L) is one of the following types.

(D.1) $(X, L) = (\mathbb{P}_{\mathbb{Q}^2}(\mathcal{F}), H(\mathcal{F}))$ and $g(X, L) = 4$, where $\mathcal{F}$ is a locally free sheaf of rank two on $\mathbb{Q}^2$.

(D.2) (X, L) is a hyperquadric fibration over $\mathbb{P}^1$, $g(X, L) = 4$ and $n = 3, 4, 5$.

(D.3) X is the Segre embedding of $\mathbb{P}^1 \times Y$ in $\mathbb{P}^7$ and $g(X, L) = 4$, where Y is a cubic surface in $\mathbb{P}^3$.

(D.4) The reduction (M, A) of (X, L) is $(\mathbb{Q}^3, \mathcal{O}_{\mathbb{Q}^3}(2))$ and $g(X, L) = 5$.

(D.5) (X, L) is a scroll over $\mathbb{P}^2$ with five double points blown up, $g(X, L) = 5$ and $n = 3$.

(D.6) (X, L) is a scroll over the first Hirzebruch surface F_1, $g(X, L) = 5$ and $n = 3$.

(D.7) X is a blowing up of a Fano manifold Y at a point in $\mathbb{P}^7$, $g(X, L) = 6$ and $n = 3$.

(D.8) X is a hypercubic section of a cone over the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^2$ in $\mathbb{P}^5$, $g(X, L) = 7$ and $n = 3$.

(D.9) (X, L) is a complete intersection of type $(3, 3)$ and $g(X, L) = 10$.

(D.10) $n = 3$, X is linked to a $\mathbb{P}^3$ in the complete intersection of a quadric and a quintic hypersurface, and $g(X, L) = 12$.

(D.11) $n = 3$, X is linked to a cubic scroll in the complete intersection of a cubic and a quartic hypersurface, and $g(X, L) = 9$.

(D.12) $n = 3$, X is a $\mathbb{P}^1$-bundle over a minimal K3 surface and L is the tautological line bundle with $g(X, L) = 8$.

(D.13) X is a hypersurface of degree 9 in $\mathbb{P}^{n+1}$ and $g(X, L) = 28$.

(D.I) First we consider the cases (D.9) and (D.13). These cases do not occur because $\mathrm{Pic}(X) \not\cong \mathbb{Z}$.

(D.II) Next we consider the case (D.1). In this case we have $\mathrm{Pic}(X) \cong \mathbb{Z}^{\oplus 3}$. Hence $\mathrm{Pic}(S) \cong \mathbb{Z}^{\oplus 2}$. By the same argument as the proof of Claim 2.3.3, we see that $\kappa(S) = -\infty$. Therefore S is a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$. We also infer that $\mathrm{rank}\mathcal{E} = 2$ because $\dim X = 3$. So we see from [20, Corollary (2.11)] that $(S, \mathcal{E})$ is one of the following.

- $S \cong \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathcal{E} \cong (p_1^*\mathcal{O}_{\mathbb{P}^2}(1) \otimes p_2^*\mathcal{O}_{\mathbb{P}^2}(1)) \oplus (p_1^*\mathcal{O}_{\mathbb{P}^2}(1) \otimes p_2^*\mathcal{O}_{\mathbb{P}^2}(2))$, where p_i is the ith projection.
- S is the blowing up of $\mathbb{P}^2$ at a point and $\mathcal{E} = (p^*(\mathcal{O}_{\mathbb{P}^2}(2)) - E)^{\oplus 2}$, where $p : S \to \mathbb{P}^2$ is the morphism and E is the exceptional divisor of p.

But here we assume that $g(X, L) \geq 4$, so these cases do not occur.

(D.III) Next we consider the case (D.3). First we note the following.

Claim 2.3.4 $\kappa(S) = -\infty$.

Proof. Let $p : X \to \mathbb{P}^1$ be the projection map. If $\pi_{F_p} : F_p \to S$ is finite for a fiber F_p of p, then $\kappa(S) = -\infty$ because $\kappa(F_p) = -\infty$. If $\pi_{F_p} : F_p \to S$ is not finite for any fiber F_p of p, then there exists a fiber F_π of π such that $p(F_\pi)$ is a point. So by [1, Lemma 4.1.13] there exists a surjective morphism $r : S \to \mathbb{P}^1$ such that $p = r \circ \pi$. Since the irregularity of a general fiber of p is zero, so is the irregularity of a general fiber of r. Therefore $\kappa(S) = -\infty$. □

In this case we have $\mathrm{Pic}(X) \cong \mathbb{Z}^{\oplus 8}$. Hence $\mathrm{Pic}(S) \cong \mathbb{Z}^{\oplus 7}$. Since $h^1(\mathcal{O}_S) = 0$, we have $K_S^2 = 3$. On the other hand we have $g(S, c_1(\mathcal{E})) = g(X, L) = 4$ and $c_1(\mathcal{E})^2 = H(\mathcal{E})^3 + c_2(\mathcal{E}) = 12$. Hence $K_S c_1(\mathcal{E}) = -6$. Hence we have $(K_S c_1(\mathcal{E}))^2 = 36 = (K_S^2)(c_1(\mathcal{E})^2)$. By the Hodge index theorem we have $c_1(\mathcal{E}) \equiv -2K_S$, that is, S is a Del Pezzo surface of degree 3. Since $\mathrm{rank}\mathcal{E} = 2$, we see from [20, Corollary (3.14)] that $\mathcal{E} \cong \mathcal{O}(-K_S)^{\oplus 2}$. This is the type (vi) in Theorem 2.3.1.

(D.IV) Next we consider the case (D.4). Let $\mu : X \to \mathbb{Q}^3$ be the reduction map. Then μ is not an identity map because $L^3 = 9$ and $\mathcal{O}_{\mathbb{Q}^3}(2)^3 = 16$. Hence there exists an effective divisor E on X such that $E \cong \mathbb{P}^2$. If $\pi(E) \neq S$, then $\pi(E)$ is a point. But this is impossible because π is a $\mathbb{P}^1$-bundle. Hence $\pi(E) = S$ holds. Moreover $\pi_E : E \to S$ is finite because $E \cong \mathbb{P}^2$. Hence we see that $\kappa(S) = -\infty$ and $h^1(\mathcal{O}_S) = 0$. Here we prove the following.

Claim 2.3.5 $S \cong \mathbb{P}^2$.

Proof. Assume that $S \not\cong \mathbb{P}^2$. Then there exists a surjective morphism $p : S \to \mathbb{P}^1$. Hence $p \circ \pi_E : E \to \mathbb{P}^1$ is surjective. But this is impossible because $E \cong \mathbb{P}^2$. □

Therefore we see that $K_S^2 = 9$. We also have $c_1(\mathcal{E})^2 = 12$ and $g(S, c_1(\mathcal{E})) = g(X, L) = 5$. Therefore $K_S c_1(\mathcal{E}) = -4$. But this is impossible because of the Hodge index theorem.

(D.V) Next we consider the case (D.6). By the same argument as the proof of Claim 2.3.3, we have $\kappa(S) = -\infty$.

In this case we have $\mathrm{Pic}(X) \cong \mathbb{Z}^{\oplus 3}$. Hence $\mathrm{Pic}(S) \cong \mathbb{Z}^{\oplus 2}$. Since $h^1(\mathcal{O}_S) = 0$, we have $K_S^2 = 8$. On the other hand we have $g(S, c_1(\mathcal{E})) = g(X, L) = 5$ and $c_1(\mathcal{E})^2 = 12$. Hence $K_S c_1(\mathcal{E}) = -4$. But this is impossible because of the Hodge index theorem.

(D.VI) Next we consider the case (D.7). In this case there exists an effective divisor E on X such that $E \cong \mathbb{P}^2$. Then we see that $\pi_E : E \to S$ is finite, $\kappa(S) = -\infty$ and $h^1(\mathcal{O}_S) = 0$ by the same reason as the case (D.4). By the same argument as the proof of Claim 2.3.5 we see that $S \cong \mathbb{P}^2$. Therefore we have $K_S^2 = 9$. We also have $c_1(\mathcal{E})^2 = 12$ and $g(S, c_1(\mathcal{E})) = g(X, L) = 6$. Therefore $K_S c_1(\mathcal{E}) = -2$. But this is impossible because of the Hodge index theorem.

(D.VII) Next we consider the case (D.8). Then by the proof of [5, Proposition (2.5)], there exists a Del Pezzo fibration $f : X \to \mathbb{P}^1$. In particular $K_X + L$ is nef.

Claim 2.3.6 *$\kappa(S) = -\infty$ holds.*

Proof. Let F_f be a fiber of f. If $\pi(F_f) \neq S$ for a general fiber F_f of f, then F_f contains a fiber of π and by [1, Lemma 4.1.13] there exists a morphism $\delta : S \to \mathbb{P}^1$ such that $f = \delta \circ \pi$. Since the irregularity of a general fiber of f is 0, we see that any general fiber of δ is $\mathbb{P}^1$. Hence we get the assertion. So we may assume that $\pi(F_f) = S$ for any general fiber F_f of f. If $\pi_{F_f} : F_f \to S$ is not a finite morphism, then F_f contains a fiber of π and we get the assertion by the same argument as above. So we may assume that $\pi_{F_f} : F_f \to S$ is a finite morphism. Since $\kappa(F_f) = -\infty$, we have $\kappa(S) = -\infty$. □

By taking a general member D of $|L|$, D is a smooth projective surface and $\kappa(D) \geq 0$ because $K_X + L$ is nef. But since $\pi_D : D \to S$ is birational, this is a contradiction.

(D.VIII) Next we consider the case (D.10). By the case 8) in Table I of [2, Page 53] we have $\kappa(X) = 1$. But this is impossible.

(D.IX) Next we consider the case (D.11). Let $D \in |L|$ be a general member. Then D is a smooth projective surface and $\pi_D : D \to S$ is birational. Hence $\chi(\mathcal{O}_D) = \chi(\mathcal{O}_S)$. By the case 9) in Table I of [2, Page 53], we have $\chi(\mathcal{O}_D) = 4$. On the other hand since $h^i(\mathcal{O}_X) = h^i(\mathcal{O}_S)$, we have $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_S) = 4$. But this is impossible because $\chi(\mathcal{O}_X) = 1$ for the case 9) in Table I of [2, Page 53].

(D.X) Next we consider the case (D.2). Let $f : X \to \mathbb{P}^1$ be the fibration. If $n \geq 4$, then $\pi(F_f)$ is a point for a general fiber F_f of f because $\mathrm{Pic}(F_f) \cong \mathbb{Z}$. Hence by [1, Lemma 4.1.13] there exists a morphism $\delta : \mathbb{P}^1 \to S$ such that $\pi = \delta \circ f$. But this is impossible because π is surjective and $\dim S = 2$. So we may assume that $n = 3$. Let $F_f = aH(\mathcal{E}) + \pi^*(B)$, where $B \in \mathrm{Pic}(S)$. Then we have

$$0 = F_f^3 = 9a^3 + 3a^2 c_1(\mathcal{E})B + 3aB^2, \quad (1)$$

$$0 = LF_f^2 = 9a^2 + 2ac_1(\mathcal{E})B + B^2, \quad (2)$$

$$2 = L^2 F_f = 9a + c_1(\mathcal{E})B. \quad (3)$$

By (1) and (2) we get $a^2 c_1(\mathcal{E})B + 2aB^2 = 0$.

If $a \neq 0$, then $B^2 = -\frac{a}{2} c_1(\mathcal{E})B$. Hence by (2) we have $c_1(\mathcal{E})B = -6a$. Therefore by (3) we get $2 = 9a + c_1(\mathcal{E})B = 3a$. But this is impossible because a is an integer. Hence $a = 0$ and $F_f = \pi^*(B)$. In particular a fiber of π is contained in a fiber of f. So by [1, Lemma 4.1.13] there exists a morphism $h : S \to \mathbb{P}^1$ such that $f = h \circ \pi$. Since $h^1(\mathcal{O}_{F_f}) = 0$, we see that $h^1(\mathcal{O}_{F_h}) = 0$ for any general fiber F_h of h. So we infer that any general fiber of h is $\mathbb{P}^1$. We note that $B = F_h$ for a fiber F_h of h. In particular we see from (3) that $F_h c_1(\mathcal{E}) = 2$ for any fiber F_h of h. On the other hand since $\mathcal{E}$ is an ample vector bundle of rank two, we infer that any fiber of h is $\mathbb{P}^1$ and therefore S is relatively minimal and S is a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$. Let C_0 be the minimal section and let $e := -C_0^2$. Since $F_h c_1(\mathcal{E}) = 2$, we can write $c_1(\mathcal{E})$ as $c_1(\mathcal{E}) \equiv 2C_0 + bF_h$. Hence $c_1(\mathcal{E})^2 = 4(b - e)$. On the other hand $c_1(\mathcal{E})^2 = H(\mathcal{E})^3 + c_2(\mathcal{E}) = 12$. So we get $b - e = 3$. Since $c_1(\mathcal{E})$ is ample, by [16, Theorem 2.12 and Corollary 2.18 in Chapter V] we have $e \geq 0$ and $b > 2e$. Therefore $3 = b - e > 2e - e = e \geq 0$, namely we get $(b, e) = (3, 0), (4, 1), (5, 2)$. We also note that $2 \leq c_1(\mathcal{E})C_0$ because $C_0 \cong \mathbb{P}^1$. Hence $2 \leq c_1(\mathcal{E})C_0 = -2e + b$ and $(b, e) = (5, 2)$ is impossible. So by Ishihara's result [20, Corollary (2.11)] we have

- $S \cong \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathcal{E} \cong (p_1^*\mathcal{O}_{\mathbb{P}^2}(1) \otimes p_2^*\mathcal{O}_{\mathbb{P}^2}(1)) \oplus (p_1^*\mathcal{O}_{\mathbb{P}^2}(1) \otimes p_2^*\mathcal{O}_{\mathbb{P}^2}(2))$, where p_i is the ith projection.
- S is a blowing up of $\mathbb{P}^2$ at a point and $\mathcal{E} = (p^*(\mathcal{O}_{\mathbb{P}^2}(2)) - E)^{\oplus 2}$, where $p : S \to \mathbb{P}^2$ is the morphism and E is the exceptional divisor of p.

But we see that $g(X, L) \leq 3$ in these cases, and these cases are ruled out.

(D.XI) Next we consider the case (D.5). By the same argument as the proof of Claim 2.3.3, we have $\kappa(S) = -\infty$.

In this case we have $\mathrm{Pic}(X) \cong \mathbb{Z}^{\oplus 7}$. Hence $\mathrm{Pic}(S) \cong \mathbb{Z}^{\oplus 6}$. Since $h^1(\mathcal{O}_S) = 0$, we have $K_S^2 = 4$. On the other hand we have $g(S, c_1(\mathcal{E})) = g(X, L) = 5$ and $c_1(\mathcal{E})^2 = H(\mathcal{E})^3 + c_2(\mathcal{E}) = 12$. Hence $K_S c_1(\mathcal{E}) = -4$. But this is impossible because of the Hodge index theorem.

(D.XII) Finally we consider the case (D.12). Let $p : X \to Y$ be the projection, where Y is a minimal K3 surface. Then there exists a very ample line bundle H on Y and a smooth member $B \in |H|$ such that $g(B) \geq 2$ and $p^*(B) =: V$ is a smooth projective surface with $\kappa(V) = -\infty$.
(i) Assume that $\pi_V : V \to S$ is surjective. Then by the same argument as the proof of Claim 2.3.3, we have $\kappa(S) = -\infty$. We note that $h^1(\mathcal{O}_S) = 0$.

If $S \cong \mathbb{P}^2$, then since $\mathrm{rank}\mathcal{E} = 2$ we see from [20, Corollary (4.7)] that $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(3)$ or $T_{\mathbb{P}^2}$. But in these cases we have $g(X, L) = g(S, c_1(\mathcal{E})) \leq 3$ and this contradicts the assumption.

If $S \not\cong \mathbb{P}^2$, then there exists a surjective morphism $h : S \to \mathbb{P}^1$ such that any general fiber of h is $\mathbb{P}^1$. Let F be a general fiber of $h \circ \pi$. If $p_F : F \to Y$ is not finite, then there exists a fiber F_p of p such that F_p is contained in F. Then by [1, Lemma 4.1.13] there exists a morphism $g : Y \to \mathbb{P}^1$ such that $g \circ p = h \circ \pi$. But since Y is a minimal K3 surface, we infer that p is an elliptic fibration and this is impossible because any general fiber of h is $\mathbb{P}^1$. Therefore $p_F : F \to Y$ is finite. But this is impossible because $\kappa(F) = -\infty$ and $\kappa(Y) = 0$.
(ii) Assume that $\pi_V : V \to S$ is not surjective. Then there exists a fiber F_π of π such that F_π is contained in V. Moreover $p(F_\pi)$ is a point because $g(B) \geq 2$ and $F_\pi \cong \mathbb{P}^1$. So by [1, Lemma 4.1.13] there exists a morphism $r : S \to Y$ such that $p = r \circ \pi$. Furthermore since p and π have connected fibers, we see that r is birational. Since p and π are $\mathbb{P}^1$-bundles, we see that r is finite. Hence r is an isomorphism and S is a minimal K3 surface. Since

$$8 = g(X, L) = g(S, c_1(\mathcal{E})) = 1 + \frac{c_1(\mathcal{E})^2}{2},$$

we have $c_1(\mathcal{E})^2 = 14$. Therefore $c_2(\mathcal{E}) = c_1(\mathcal{E})^2 - H(\mathcal{E})^3 = 14 - 9 = 5$, and this is impossible.

□

3 Main Theorem

Theorem 3.1 *Let (X, L) be a polarized manifold of dimension $n \geq 3$ and let (M, A) be a reduction of (X, L). Assume that L is very ample. If $b_2(X, L) = h^2(X, \mathbb{C}) + 2$, then (X, L) is one of the* *following types.*

(i) $(\mathbb{P}_S(\mathcal{E}), H(\mathcal{E}))$, *where* S *is a smooth projective surface and* $\mathcal{E}$ *is a very ample vector bundle on* S *with* $c_2(\mathcal{E}) = 3$. *In particular* $(S, \mathcal{E})$ *is one of the following.*

  (i.1) $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 3})$.

  (i.2) $(\mathbb{P}^2, T_{\mathbb{P}^2})$, *where* $T_{\mathbb{P}^2}$ *is the tangent bundle of* $\mathbb{P}^2$.

  (i.3) $(\mathbb{P}^1 \times \mathbb{P}^1, [p_1^*(\mathcal{O}_{\mathbb{P}^2}(1)) \otimes p_2^*(\mathcal{O}_{\mathbb{P}^2}(2))] \oplus [p_1^*(\mathcal{O}_{\mathbb{P}^2}(1)) \otimes p_2^*(\mathcal{O}_{\mathbb{P}^2}(1))])$.

  (i.4) S *is a blowing up of* $\mathbb{P}^2$ *at a point and* $\mathcal{E} = (p^*(\mathcal{O}_{\mathbb{P}^2}(2)) - E)^{\oplus 2}$, *where* $p : S \to \mathbb{P}^2$ *is the birational morphism and* E *is the exceptional divisor of* p.

  (i.5) $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(3))$.

  (i.6) S *is a Del Pezzo surface of degree* 3 *and* $\mathcal{E} \cong \mathcal{O}(-K_S)^{\oplus 2}$.

(ii) (M, A) *is a Del Pezzo fibration over a smooth curve* C *with* $n = 3, 4$. *Let* $f : M \to C$ *be its morphism. In this case there exists an ample line bundle* H *on* C *such that* $K_M + (n-2)A = f^*(H)$, *and we have* $(g(C), \deg H) = (1, 1)$, $b_2(M, A) = 14$ *and* $h^2(M, \mathbb{C}) = 12$.

(iii) (M, A) *is a quadric fibration over a smooth surface* S *with* $n = 3, 4$. *Let* $f : M \to S$ *be its morphism. In this case there exists an ample line bundle* H *on* S *such that* $K_M + (n-2)A = f^*(K_S + H)$, *and* (S, H) *is one of the following types:*

  (iii.1) S *is a* $\mathbb{P}^1$*-bundle,* $p : S \to B$, *over a smooth elliptic curve* B, *and* $H = 3C_0 - F$, *where* C_0 (*resp.* F) *denotes the minimal section of* S *with* $C_0^2 = 1$ (*resp. a fiber of* p). *In this case* $b_2(M, A) = 12$ *and* $h^2(M, \mathbb{C}) = 10$.

  (iii.2) S *is an abelian surface,* $H^2 = 2$, *and* $h^0(H) = 1$. *In this case* $b_2(M, A) = 14$ *and* $h^2(M, \mathbb{C}) = 12$.

  (iii.3) S *is a hyperelliptic surface,* $H^2 = 2$, *and* $h^0(H) = 1$. *In this case* $b_2(M, A) = 10$ *and* $h^2(M, \mathbb{C}) = 8$.

Proof. First we note that the following hold.

- $b_2(X, L) = 2g_2(X, L) + h_2^{1,1}(X, L)$ by Remark 2.1.1 (ii.1) and (ii.3).
- $g_2(X, L) \geq h^2(\mathcal{O}_X)$ by Remark 2.1.1 (iii.3).
- $h_2^{1,1}(X, L) \geq h^{1,1}(X)$ by Remark 2.1.1 (iii.2).
- $h^2(X, \mathbb{C}) = 2h^2(\mathcal{O}_X) + h^{1,1}(X)$ by the Hodge theory.

Hence we see from $b_2(X, L) = h^2(X, \mathbb{C}) + 2$ that one of the following holds.

(A) $g_2(X, L) = h^2(\mathcal{O}_X)$ and $h_2^{1,1}(X, L) = h^{1,1}(X) + 2$.

(B) $g_2(X, L) = h^2(\mathcal{O}_X) + 1$ and $h_2^{1,1}(X, L) = h^{1,1}(X)$.

(A) First we consider the case (A). Since L is very ample and $g_2(X, L) = h^2(\mathcal{O}_X)$, by [10, Corollary 3.5] we infer that (X, L) is one of the types from (1) to (7.4) in Theorem 2.2.1. Since $b_2(X, L) = h^2(X, \mathbb{C}) + 2$, by using [12, Example 3.1], we see that (X, L) is one of the following types as possibility.

(a) $(\mathbb{P}^2 \times \mathbb{P}^2, \otimes_{i=1}^2 p_i^* \mathcal{O}_{\mathbb{P}^2}(1))$, where p_i is the ith projection.

(b) $(\mathbb{P}_{\mathbb{P}^2}(T_{\mathbb{P}^2}), H(T_{\mathbb{P}^2}))$, where $T_{\mathbb{P}^2}$ is the tangent bundle of $\mathbb{P}^2$.

(c) A hyperquadric fibration over a smooth curve.

(d) $(\mathbb{P}_S(\mathcal{E}), H(\mathcal{E}))$, where S is a smooth projective surface and $\mathcal{E}$ is a very ample vector bundle on S with $c_2(\mathcal{E}) = 3$.

(e) A reduction (M, A) of (X, L) is a Veronese fibration over a smooth curve C, that is, M is a $\mathbb{P}^2$-bundle over C and $A|_F = \mathcal{O}_{\mathbb{P}^2}(2)$ for every fiber F of it.

(A.1) If (X, L) is one of the types (a) and (b), then we see from [12, Example 3.1] that $b_2(X, L) = h^2(X, \mathbb{C}) + 2$. The case (a) (resp. (b)) corresponds to the case (i.1) (resp. (i.2)) in Theorem 3.1.

(A.2) Next we consider the case (c) and we use notation in Notation 2.2.1 (1). Here we note that $h^2(X, \mathbb{C}) = 2$ in this case (see Definition 2.2.1 (3)). Since $b_2(X, L) = h^2(X, \mathbb{C}) + 2$ and $h^2(X, \mathbb{C}) = 2$, we see from [12, Example 3.1 (5)] that $2e + 3b = 2$. On the other hand, from the fact that $L^n = 2e + b > 0$ and $2e + (n + 1)b \geq 0$ by [7, (3.3)], we get the following.

Claim 3.1 *$(e, b) = (1, 0)$ or $(4, -2)$. Moreover $n = 3$ if $(e, b) = (4, -2)$.*

Proof. If $b > 0$, then $2e+3b = 2e+b+2b \geq 3$ and this is impossible. So we have $b \leq 0$. If $b = 0$, then $e = 1$. So we assume that $b < 0$. Then $2 = 2e+3b = 2e+(n+1)b-(n-2)b \geq -(n-2)b \geq -b$ because $n \geq 3$ and $b < 0$. So we have $b = -2$ or -1. If $b = -1$, then $2 = 2e + 3b = 2e - 3$. But this is impossible because e is an integer. Hence $b = -2$ and we see from the above inequality that $n = 3$. We also note that $2e + 3b = 2$ implies $e = 4$. □

(A.2.1) If $(e, b) = (1, 0)$, then $L^n = 2e + b = 2$. Therefore we see that $(X, L) \cong (\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1))$ because L is very ample. Since $n \geq 3$, we have $\mathrm{Pic}(X) \cong \mathbb{Z}$. But this is impossible because (X, L) is a hyperquadric fibration over a smooth curve.
(A.2.2) Assume that $(e, b) = (4, -2)$. In this case $n = 3$ by Claim 3.1. Therefore $\mathrm{rank}\mathcal{E} = 4$. On the other hand we see from $L^3 = 6$ that $h^1(\mathcal{O}_X) = 0$ holds by Ionescu's result [17]. Hence $C = \mathbb{P}^1$. Therefore by the Riemann-Roch theorem we have

$$h^0(L) = h^0(\mathcal{E}) = \deg \mathcal{E} + (\mathrm{rank}\mathcal{E})\, \chi(\mathcal{O}_C) = 8$$

and X is embedded in $\mathbb{P}^7$. We see from the list of [17] that (X, L) is a Del Pezzo manifold, but this is impossible because $\mathcal{O}(K_X + (n - 1)L) \neq \mathcal{O}_X$ in this case.

(A.3) Next we consider the case (e). We use notation in Notation 2.2.1 (2). From [12, Example 3.1 (7.4)] we have

$$2e + 3b = 2 \tag{4}$$

because $b_2(X, L) = h^2(X, \mathbb{C}) + 2e + 3b$. Here we note that by [12, Remark 2.6]

$$g_1(M, A) = 2e + 2b + 1. \tag{5}$$

We also note that $g_1(M, A) \geq 2$ in this case because $K_M + 2A$ is ample. Hence by (5) we have

$$2e + 2b \geq 1. \tag{6}$$

Moreover by [12, Remark 2.6]

$$e + 2b + 2g(C) - 2 = 0. \tag{7}$$

Hence we see from (4) and (7)

$$b = 2 - 4g(C), \tag{8}$$
$$e = 6g(C) - 2. \tag{9}$$

By (6), (8) and (9), we get $2g(C) = b + e \geq \frac{1}{2}$, that is, $g(C) \geq 1$.

Then we have $L^3 \leq A^3 = 8e + 12b = 8$. Since L is very ample and $n = 3$, we have $h^0(L) \geq 4$. Assume that $h^0(L) = 4$. Then X is a 3-dimensional projective space. But this is impossible because X is a fiber space over a smooth curve. Next we consider the case $h^0(L) = 5$. Then X is a hypersurface in $\mathbb{P}^4$ and we have $\mathrm{Pic}(X) \cong \mathbb{Z}$ in this case. But this is also impossible. So we may assume that $h^0(L) \geq 6$.
(A.3.1) If $L^3 \leq 5$, then $L^3 \geq 2\Delta(X, L) + 1$ and $g(X, L) \geq 2 \geq L^3 - 3 \geq \Delta(X, L)$. Hence we see from [9, (3.5) Theorem] that $h^1(\mathcal{O}_X) = 0$. But this is a contradiction because $g(C) \geq 1$.
(A.3.2) Assume that $L^3 = 6$. If $h^0(L) \geq 7$, then $L^3 = 6 > 5 \geq 2\Delta(X, L) + 1$ and $g(X, L) \geq 2 \geq \Delta(X, L)$. Hence we see from [9, (3.5) Theorem] that $h^1(\mathcal{O}_X) = 0$. But this is a contradiction.

If $h^0(L) = 6$, then X is embedded in $\mathbb{P}^5$ and by Ionescu's result [17] we have $h^1(\mathcal{O}_X) = 0$. But this is a contradiction.
(A.3.3) Assume that $L^3 = 7$. If $h^0(L) \geq 7$, then $L^3 = 7 \geq 2\Delta(X, L) + 1$ and $g(X, L) = g(M, A) = 2e + 2b + 1 = 4g(C) + 1 \geq 5 > 3 \geq \Delta(X, L)$. Hence we see from [9, (3.5) Theorem] that $h^1(\mathcal{O}_X) = 0$. But this is a contradiction.

If $h^0(L) = 6$, then X is embedded in $\mathbb{P}^5$ and by Ionescu's result [17] we have $h^1(\mathcal{O}_X) = 0$. But this is a contradiction.
(A.3.4) Assume that $L^3 = 8$. If $h^0(L) \geq 8$, then $L^3 = 8 > 7 \geq 2\Delta(X, L) + 1$ and $g(X, L) = g(M, A) = 2e + 2b + 1 = 4g(C) + 1 \geq 5 > 3 \geq \Delta(X, L)$. Hence we see from [9, (3.5) Theorem] that $h^1(\mathcal{O}_X) = 0$. But this is a contradiction.

If $h^0(L) = 7$ (resp. 6), then X is embedded in $\mathbb{P}^6$ (resp. $\mathbb{P}^5$) and by Ionescu's result [19] we have $h^1(\mathcal{O}_X) = 0$. But this is a contradiction.

(A.4) Next we consider the case (d). In this case, since $\mathcal{E}$ is a very ample vector bundle with $c_2(\mathcal{E}) = 3$, we see from Theorem 2.3.1 that $(S, \mathcal{E})$ is one of the types from (i.1) to (i.6) in Theorem 3.1.

(B) Next we consider the case (B). Let (M, A) be a reduction of (X, L). Since L is very ample and $g_2(X, L) = h^2(\mathcal{O}_X) + 1$, by [10, Theorem 3.6] and [13, Theorem 1] we infer that (X, L) is one of the following types.

- (f) (M, A) is a Mukai manifold.
- (g) (M, A) is a Del Pezzo fibration over a smooth curve C. Let $f : M \to C$ be its morphism. In this case there exists an ample line bundle H on C such that $K_M + (n-2)A = f^*(H)$ and $(g(C), \deg H) = (1, 1)$.
- (h) (M, A) is a quadric fibration over a smooth surface S. Let $f : M \to S$ be its morphism. In this case there exists an ample line bundle H on S such that $K_M + (n-2)A = f^*(K_S + H)$ and (S, H) is one of the following types:
  - (h.1) S is a $\mathbb{P}^1$-bundle, $p : S \to B$, over a smooth elliptic curve B, and $H = 3C_0 - F$, where C_0 (resp. F) denotes the minimal section of S with $C_0^2 = 1$ (resp. a fiber of p).
  - (h.2) S is an abelian surface, $H^2 = 2$, and $h^0(H) = 1$.
  - (h.3) S is a hyperelliptic surface, $H^2 = 2$, and $h^0(H) = 1$.

First we note that $b_2(X, L) - h^2(X, \mathbb{C}) = b_2(M, A) - h^2(M, \mathbb{C})$ by [12, Remark 2.2 (3)].

(B.1) First we consider the case (f). Then we see from [10, Example 2.10 (7)] that $(K_M + (n-2)A)^2 A^{n-2} = 0$, $h^1(\mathcal{O}_M) = 0$ and $g_2(M, A) = 1$ holds. Hence by [12, Proposition 3.1] we have

$$h_2^{1,1}(M, A) = 10(1 - h^1(\mathcal{O}_M) + g_2(M, A)) - (K_M + (n-2)A)^2 A^{n-2} + 2h^1(\mathcal{O}_M) = 20.$$

Therefore $b_2(M, A) = 2g_2(M, A) + h_2^{1,1}(M, A) = 22$.

Next we calculate $h^2(M,\mathbb{C})$. Since L is very ample, there exist $n-3$ members $D_1,\dots,D_{n-3}$ of $|A|$ such that $M_{n-3} := D_1 \cap \dots \cap D_{n-3}$ is a smooth projective variety of dimension 3 and $\mathcal{O}(K_{M_{n-3}} + A_{M_{n-3}}) = \mathcal{O}_{M_{n-3}}$. By a classification of 3-dimensional Fano manifolds (see [21]), we see that $h^2(M_{n-3},\mathbb{C}) \le 10$ and by the Lefschetz theorem we get $h^2(M,\mathbb{C}) \le 10$. Therefore $b_2(X,L) - h^2(X,\mathbb{C}) = b_2(M,A) - h^2(M,\mathbb{C}) > 2$ and this case is ruled out.

(B.2) Next we consider the case (g). We note that $g_2(M,A) = 1$, $h^1(\mathcal{O}_M) = 1$ and $(K_M + (n-2)A)^2A^{n-2} = 0$ in this case. Hence by [12, Proposition 3.1] we have

$$h_2^{1,1}(M,A) = 10(1 - h^1(\mathcal{O}_M) + g_2(M,A)) - (K_M + (n-2)A)^2A^{n-2} + 2h^1(\mathcal{O}_M) = 12.$$

Therefore $b_2(M,A) = 2g_2(M,A) + h_2^{1,1}(M,A) = 14$.

Next we calculate $h^2(M,\mathbb{C})$. First we note that $\tau(A) = n-2$ in this case, where $\tau(A)$ is the nef value of A. Assume that $n \ge 5$. Then

$$\tau(A) = n - 2 > \frac{n}{2} = \frac{n - \dim C + 1}{2}.$$

Hence by the proof of [3, (3.1.1) Theorem] we see that there exists a non-breaking dominating family T of lines relative to A such that for any $t \in T$ the curve l_t corresponding to t satisfies $(K_M + (n-2)A)l_t = 0$.

(B.2.1) If $n \ge 6$, then $\tau(A) = n-2 \ge \frac{n}{2} + 1$ holds. Hence by (3.1.1.2) in [3, (3.1.1) Theorem] we see that f is an elementary contraction because $\dim C = 1$. In particular $\rho(M) = \rho(C)+1 = 2$ and we get $h^2(M,\mathbb{C}) = 2$, where $\rho(M)$ (resp. $\rho(C)$) is the Picard number of M (resp. C). Therefore $b_2(X,L) - h^2(X,\mathbb{C}) = b_2(M,A) - h^2(M,\mathbb{C}) > 2$ and this case is ruled out.

(B.2.2) Next we consider the case $n = 5$. Let l be a line on M relative to A such that l is the curve corresponding to a point of T and let $\nu := -K_M l - 2$. Since $(K_M + (n-2)A)l = 0$, we have $-K_M l = 3$. Hence $\nu = 1$. On the other hand $\tau(A) = n-2 = 3$. So we get $\nu = 1 \ge 1 = \frac{n-3}{2}$ and $\nu = 1 = \tau(A) - 2$. Hence by [3, (2.5) Theorem] we see that either (2.5.1) or (2.5.2) in [3, (2.5) Theorem] holds because $\dim C = 1$.

If (2.5.1) in [3, (2.5) Theorem] holds, then f is an elementary contraction and $\rho(M) = \rho(C)+1 = 2$. So we get $h^2(M,\mathbb{C}) = 2$. Therefore $b_2(X,L) - h^2(X,\mathbb{C}) = b_2(M,A) - h^2(M,\mathbb{C}) > 2$ and this case is ruled out.

If (2.5.2) in [3, (2.5) Theorem] holds, then there exist two morphism $\phi : M \to W$ and $\pi : W \to C$ such that ϕ is a $\mathbb{P}^2$-bundle over a smooth projective variety W of dimension 3, π is a $\mathbb{P}^2$-bundle over C and $f = \pi \circ \phi$. In this case $\rho(M) = \rho(W) + 1 = \rho(C) + 2 = 3$. So we get $h^2(M,\mathbb{C}) = 3$. Therefore $b_2(X,L) - h^2(X,\mathbb{C}) = b_2(M,A) - h^2(M,\mathbb{C}) > 2$ and this case is ruled out.

(B.3) Finally we consider the case (h). In this case, $g_2(M,A) = h^2(\mathcal{O}_M) + 1 = h^2(\mathcal{O}_S) + 1$ and $(K_M + (n-2)A)^2A^{n-2} = 2(K_S + H)^2$. So we get

$$\begin{aligned} h_2^{1,1}(M,A) &= 10(1 - h^1(\mathcal{O}_M) + g_2(M,A)) - (K_M + (n-2)A)^2A^{n-2} + 2h^1(\mathcal{O}_M) \\ &= 10(\chi(\mathcal{O}_S) + 1) - 2(K_S + H)^2 + 2h^1(\mathcal{O}_S). \end{aligned}$$

(B.3.1) We consider the case (h.1). Then $(K_S+H)^2 = 1$, $h^2(S,\mathbb{C}) = 2$, $h^1(\mathcal{O}_S) = 1$ and $h^2(\mathcal{O}_S) = 0$. Hence $g_2(M,A) = 1$, $h_2^{1,1}(M,A) = 10$ and $b_2(M,A) = 2g_2(M,A) + h_2^{1,1}(M,A) = 12$.

(B.3.2) We consider the case (h.2). Then $(K_S+H)^2 = 2$, $h^2(S,\mathbb{C}) = 6$, $h^1(\mathcal{O}_S) = 2$ and $h^2(\mathcal{O}_S) = 1$. Hence $g_2(M,A) = 2$, $h_2^{1,1}(M,A) = 10$ and $b_2(M,A) = 2g_2(M,A) + h_2^{1,1}(M,A) = 14$.

(B.3.3) We consider the case (h.3). Then $(K_S+H)^2 = 2$, $h^2(S,\mathbb{C}) = 2$, $h^1(\mathcal{O}_S) = 1$ and $h^2(\mathcal{O}_S) = 0$.

Hence $g_2(M,A)=1$, $h_2^{1,1}(M,A)=8$ and $b_2(M,A)=2g_2(M,A)+h_2^{1,1}(M,A)=10$.

Next we calculate $h^2(M,\mathbb{C})$. First we note that $\tau(A)=n-2$ in this case.
Assume that $n\geq 4$. Then

$$\tau(A)=n-2>\frac{n-1}{2}=\frac{n-\dim S+1}{2}.$$

Hence by [3, (3.1.1) Theorem] we see that there exists a non-breaking dominating family of lines relative to A such that for any $t\in T$ the curve l_t corresponding to t satisfies $(K_M+(n-2)A)l_t=0$.

If $n\geq 6$, then $\tau(A)=n-2\geq \frac{n}{2}+1$ holds. Hence by (3.1.1.2) in [3, (3.1.1) Theorem] we see that f is an elementary contraction because $\dim S=2$. In particular $\rho(M)=\rho(S)+1$ and we get $h^2(M,\mathbb{C})=h^2(S,\mathbb{C})+1$. Therefore $b_2(X,L)-h^2(X,\mathbb{C})=b_2(M,A)-h^2(M,\mathbb{C})>2$ for each case and the case where $n\geq 6$ is ruled out.

Next we consider the case $n=5$. Let l be a line on M relative to A such that l is the curve corresponding to a point of T and let $\nu:=-K_Ml-2$. Since $(K_M+(n-2)A)l=0$, we have $-K_Ml=3$. Hence $\nu=1$. On the other hand $\tau(A)=n-2=3$. So we get $\nu=1\geq 1=\frac{n-3}{2}$ and $\nu=1=\tau(A)-2$. Hence by [3, (2.5) Theorem] we see that (2.5.1) in [3, (2.5) Theorem] holds because $\dim S=2$. Then f is an elementary contraction and $\rho(M)=\rho(S)+1$. So we get $h^2(M,\mathbb{C})=h^2(S,\mathbb{C})+1$. Therefore $b_2(X,L)-h^2(X,\mathbb{C})=b_2(M,A)-h^2(M,\mathbb{C})>2$ and the case where $n=5$ is also ruled out.

Therefore we get the assertion. □

Corollary 3.1 *Let (X,L) be a polarized manifold of dimension $n\geq 3$. Assume that L is very ample. If $b_2(X,L)=h^2(X,\mathbb{C})+2$, then $n=3$ or 4.*

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