Classification of polarized manifolds by the second sectional Betti number *†‡

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Abstract

Let X be an n-dimensional smooth projective variety defined over the field of complex numbers, let L be an ample and spanned line bundle on X. Then we classify (X, L) with $b_2(X, L) = h^2(X, \mathbb{C}) + 1$, where $b_2(X, L)$ is the second sectional Betti number of (X, L).

1 Introduction

Let X be a smooth projective variety of dimension n defined over the field of complex numbers $\mathbb{C}$ and let L be an ample line bundle on X. Then we call this pair (X, L) a polarized manifold. In [6], for every integer i with $0 \leq i \leq n$, we defined the invariant $b_i(X, L)$ which is called the *ith sectional Betti number of* (X, L). If L is spanned, then we can prove that $b_i(X, L) \geq h^i(X, \mathbb{C})$ (see Remark 2.1.1 (iii.1) below). So it is interesting to classify (X, L) by the value of $b_i(X, L) - h^i(X, \mathbb{C})$.

In this paper, we consider the case of $i = 2$. Then in [7, Theorem 4.1] we have classified polarized manifolds (X, L) such that L is spanned and $b_2(X, L) = h^2(X, \mathbb{C})$.

So in this paper, as the next step, we will classify polarized manifolds (X, L) such that L is spanned and $b_2(X, L) = h^2(X, \mathbb{C}) + 1$.

In this paper we will use the customary notation in algebraic geometry.

2 Preliminaries

2.1 Review on sectional invariants of polarized manifolds

In this subsection, we will review the theory of sectional invariants of polarized manifolds which will be used in the main theorem (Theorem 3.1) and its proof.

Notation 2.1.1 (1) Let X be a projective variety of dimension n, let L be an ample line bundle on X. Then the Euler-Poincaré characteristic $\chi(L^{\otimes t})$ of $L^{\otimes t}$ is a polynomial in t of degree n, and we can describe $\chi(L^{\otimes t})$ as follows.

$$\chi(L^{\otimes t}) = \sum_{j=0}^{n} \chi_j(X, L) \binom{t + j - 1}{j}.$$

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(2) Let Y be a smooth projective variety of dimension i, let $\mathcal{T}_Y$ be the tangent bundle of Y, and let Ω_Y be the dual bundle of $\mathcal{T}_Y$. For every integer j with $0 \leq j \leq i$, we put

$$\begin{aligned} h_{i,j}(c_1(Y), \cdots, c_i(Y)) &:= \chi(\Omega_Y^j) \\ &= \int_Y \mathrm{ch}(\Omega_Y^j)\mathrm{Td}(\mathcal{T}_Y). \end{aligned}$$

(Here $\mathrm{ch}(\Omega_Y^j)$ (resp. $\mathrm{Td}(\mathcal{T}_Y)$) denotes the Chern character of Ω_Y^j (resp. the Todd class of $\mathcal{T}_Y$). See [8, Examples 3.2.3 and 3.2.4].)

(3) Let (X, L) be a polarized manifold of dimension n. For every integers i and j with $0 \leq j \leq i \leq n$, we put

$$C_j^i(X, L) := \sum_{l=0}^{j} (-1)^l \binom{n-i+l-1}{l} c_{j-l}(X)L^l,$$

$$w_i^j(X, L) := h_{i,j}(C_1^i(X, L), \cdots, C_i^i(X, L))L^{n-i}.$$

(4) Let X be a smooth projective variety of dimension n. For every integers i and j with $0 \leq j \leq i \leq n$, we put

$$H_1(i, j) := \begin{cases} \displaystyle\sum_{s=0}^{i-j-1} (-1)^s h^s(\Omega_X^j) & \text{if } j \neq i, \\ 0 & \text{if } j = i, \end{cases}$$

$$H_2(i, j) := \begin{cases} \displaystyle\sum_{t=0}^{j-1} (-1)^{i-t} h^t(\Omega_X^{i-j}) & \text{if } j \neq 0, \\ 0 & \text{if } j = 0. \end{cases}$$

Definition 2.1.1 (See [5, Definition 2.1] and [6, Definition 3.1].) Let (X, L) be a polarized manifold of dimension n, and let i and j be integers with $0 \leq i \leq n$ and $0 \leq j \leq i$. (Here we use Notation 2.1.1.)

(1) The *ith sectional geometric genus* $g_i(X, L)$ *of* (X, L) is defined as follows:

$$g_i(X, L) := (-1)^i(\chi_{n-i}(X, L) - \chi(\mathcal{O}_X)) + \sum_{j=0}^{n-i} (-1)^{n-i-j} h^{n-j}(\mathcal{O}_X).$$

(2) The *ith sectional Euler number* $e_i(X, L)$ *of* (X, L) is defined by the following:

$$e_i(X, L) := C_i^i(X, L)L^{n-i}.$$

(3) The *ith sectional Betti number* $b_i(X, L)$ *of* (X, L) is defined by the following:

$$b_i(X, L) := \begin{cases} e_0(X, L) & \text{if } i = 0, \\ (-1)^i \left(e_i(X, L) - \displaystyle\sum_{j=0}^{i-1} 2(-1)^j h^j(X, \mathbb{C}) \right) & \text{if } 1 \leq i \leq n. \end{cases}$$

(4) The *ith sectional Hodge number* $h_i^{j,i-j}(X, L)$ *of type* $(j, i-j)$ *of* (X, L) is defined by the following:

$$h_i^{j,i-j}(X, L) := (-1)^{i-j} \left\{ w_i^j(X, L) - H_1(i, j) - H_2(i, j) \right\}.$$

Remark 2.1.1 (i) For every integers i and j with $0 \leq i \leq n$ and $0 \leq j \leq i$, $g_i(X,L)$, $e_i(X,L)$, $b_i(X,L)$ and $h_i^{j,i-j}(X,L)$ are integer (see [6, Proposition 3.1]).

(ii) Let (X,L) be a polarized manifold of dimension n. For every integers i and j with $0 \leq i \leq n$ and $0 \leq j \leq i$, we get the following (see [6, Theorem 3.1]).

(ii.1) $b_i(X,L) = \sum_{k=0}^{i} h_i^{k,i-k}(X,L)$.

(ii.2) $h_i^{j,i-j}(X,L) = h_i^{i-j,j}(X,L)$.

(ii.3) $h_i^{i,0}(X,L) = h_i^{0,i}(X,L) = g_i(X,L)$.

(iii) Assume that L is ample and spanned. Then, for every integers i and j with $0 \leq i \leq n$ and $0 \leq j \leq i$, the following inequality hold (see [5, Theorem 3.1] and [6, Proposition 3.3]).

(iii.1) $b_i(X,L) \geq h^i(X,\mathbb{C})$.

(iii.2) $h_i^{j,i-j}(X,L) \geq h^{j,i-j}(X)$.

(iii.3) $g_i(X,L) \geq h^i(\mathcal{O}_X)$.

2.2 Adjunction theory of polarized manifolds

In this subsection, we will review the adjunction theory which will be used later.

Definition 2.1 Let (X,L) be a polarized manifold of dimension n.

(1) We say that (X,L) is a *scroll* (resp. *quadric fibration*) *over a normal projective variety* Y *of dimension* m with $1 \leq m < n$ if there exists a surjective morphism with connected fibers $f: X \to Y$ such that $K_X + (n-m+1)L = f^*A$ (resp. $K_X + (n-m)L = f^*A$) for some ample line bundle A on Y.

(2) (X,L) is called a *classical scroll over a normal variety* Y if there exists a vector bundle $\mathcal{E}$ on Y such that $X \cong \mathbb{P}_Y(\mathcal{E})$ and $L = H(\mathcal{E})$, where $H(\mathcal{E})$ is the tautological line bundle.

(3) We say that (X,L) is a *hyperquadric fibration over a smooth projective curve* C if (X,L) is a quadric fibration over C such that the morphism $f: X \to C$ is the contraction morphism of an extremal ray. In this case, $(F, L_F) \cong (\mathbb{Q}^{n-1}, \mathcal{O}_{\mathbb{Q}^{n-1}}(1))$ for any general fiber F of f, every fiber of f is irreducible and reduced (see [9] or [3, Claim (3.1)]) and $h^2(X,\mathbb{C}) = 2$.

Remark 2.1 (1) If (X,L) is a scroll over a smooth projective curve C, then (X,L) is a classical scroll over C (see [1, Proposition 3.2.1]).

(2) If (X,L) is a scroll over a normal projective surface S, then S is smooth and (X,L) is also a classical scroll over S (see [2, (3.2.1) Theorem] and [4, Chapter II, (11.8.6)]).

(3) Assume that (X,L) is a quadric fibration over a smooth curve C with $\dim X = n \geq 3$. Let $f: X \to C$ be its morphism. By [2, (3.2.6) Theorem] and the proof of [9, Lemma (c) in Section 1], we see that (X,L) is one of the following:

(a) A hyperquadric fibration over C.

(b) A classical scroll over a smooth surface with $\dim X = 3$.

Theorem 2.2.1 *Let* (X,L) *be a polarized manifold with* $\dim X = n \geq 3$. *Then* (X,L) *is one of the following types.*

(1) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$.

(2) $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1))$.

(3) *A scroll over a smooth projective curve.*

(4) $K_X \sim -(n-1)L$, *that is,* (X, L) *is a Del Pezzo manifold.*

(5) *A hyperquadric fibration over a smooth projective curve.*

(6) *A classical scroll over a smooth projective surface.*

(7) *Let* (M, A) *be a reduction of* (X, L).

  (7.1) $n = 4$, $(M, A) = (\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2))$.

  (7.2) $n = 3$, $(M, A) = (\mathbb{Q}^3, \mathcal{O}_{\mathbb{Q}^3}(2))$.

  (7.3) $n = 3$, $(M, A) = (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3))$.

  (7.4) $n = 3$, *M is a $\mathbb{P}^2$-bundle over a smooth curve C and* $(F', A|_{F'}) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ *for any fiber F' of it.*

  (7.5) $K_M + (n-2)A$ *is nef.*

Proof. See [1, Proposition 7.2.2, Theorems 7.2.4, 7.3.2 and 7.3.4] and [4, Chapter II, (11.2), (11.7), and (11.8)]. □

Notation 2.2.1 (1) Let (X, L) be a hyperquadric fibration over a smooth curve C and let $f : X \to C$ be its morphism. We put $\mathcal{E} := f_*(L)$. Then $\mathcal{E}$ is a locally free sheaf of rank $n+1$ on C. Let $\pi : \mathbb{P}_C(\mathcal{E}) \to C$ be the projective bundle. Then $X \in |2H(\mathcal{E}) + \pi^*(B)|$ for some $B \in \mathrm{Pic}(C)$ and $L = H(\mathcal{E})|_X$, where $H(\mathcal{E})$ is the tautological line bundle of $\mathbb{P}_C(\mathcal{E})$. We put $e := \deg \mathcal{E}$ and $b := \deg B$.

(2) (See [4, (13.10) Chapter II].) Let (M, A) be a $\mathbb{P}^2$-bundle over a smooth curve C and $A|_F = \mathcal{O}_{\mathbb{P}^2}(2)$ for any fiber F of it. Let $f : M \to C$ be the fibration and $\mathcal{E} := f_*(K_M + 2A)$. Then $\mathcal{E}$ is a locally free sheaf of rank 3 on C, and $M \cong \mathbb{P}_C(\mathcal{E})$ such that $H(\mathcal{E}) = K_M + 2A$. In this case, $A = 2H(\mathcal{E}) + f^*(B)$ for a line bundle B on C, and by the canonical bundle formula we have $K_M = -3H(\mathcal{E}) + f^*(K_C + \det\mathcal{E})$. Here we set $e := \deg \mathcal{E}$ and $b := \deg B$.

3 Main Theorem

Theorem 3.1 *Let (X, L) be a polarized manifold of dimension $n \geq 3$. Assume that L is spanned. If $b_2(X, L) = h^2(X, \mathbb{C}) + 1$, then (X, L) is one of the following types.*

(a) $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1))$.

(b) $(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2))$.

(c) *A simple blowing up of (X, L) of type* (b).

(d) $(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1, \otimes_{i=1}^3 p_i^* \mathcal{O}_{\mathbb{P}^1}(1))$, *where p_i is the ith projection.*

(e) $(\mathbb{P}_S(\mathcal{E}), H(\mathcal{E}))$, *where S is a smooth projective surface and $\mathcal{E}$ is an ample and spanned vector bundle of rank two on S with $c_2(\mathcal{E}) = 2$. In particular $(S, \mathcal{E})$ is one of the following.*

  (e.1) $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(2))$.

  (e.2) $(\mathbb{Q}^2, \mathcal{O}_{\mathbb{Q}^2}(1) \oplus \mathcal{O}_{\mathbb{Q}^2}(1))$.

  (e.3) $(\mathbb{P}_C(\mathcal{F}), \pi^*(\mathcal{G}) \otimes H(\mathcal{F}))$, *where C is an elliptic curve, $\mathcal{F}$ and $\mathcal{G}$ are indecomposable vector bundles of rank two on C with $\deg \mathcal{F} = 1$ and $\deg \mathcal{G} = 1$, and $\pi : \mathbb{P}_C(\mathcal{F}) \to C$ is the projection map.*

(e.4) *S is a double covering of $\mathbb{P}^2$, $f : S \to \mathbb{P}^2$, and $\mathcal{E} \cong f^*(\mathcal{O}_{\mathbb{P}^2}(1)) \oplus f^*(\mathcal{O}_{\mathbb{P}^2}(1))$.*

Proof. First we note that the following hold.

- $b_2(X, L) = 2g_2(X, L) + h_2^{1,1}(X, L)$ by Remark 2.1.1 (ii.1) and (ii.3).
- $g_2(X, L) \geq h^2(\mathcal{O}_X)$ by Remark 2.1.1 (iii.3).
- $h_2^{1,1}(X, L) \geq h^{1,1}(X)$ by Remark 2.1.1 (iii.2).
- $h^2(X, \mathbb{C}) = 2h^2(\mathcal{O}_X) + h^{1,1}(X)$ by the Hodge theory.

Hence we see from $b_2(X, L) = h^2(X, \mathbb{C}) + 1$ that $g_2(X, L) = h^2(\mathcal{O}_X)$ and $h_2^{1,1}(X, L) = h^{1,1}(X) + 1$. Since L is spanned and $g_2(X, L) = h^2(\mathcal{O}_X)$, by [5, Corollary 3.5] we infer that (X, L) is one of the types from (1) to (7.4) in Theorem 2.2.1. Since $b_2(X, L) = h^2(X, \mathbb{C}) + 1$, by using [7, Example 3.1], we see that (X, L) is one of the following types as possibility.

(i) $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1))$.

(ii) $(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2))$.

(iii) A simple blowing up of (X, L) of type (ii).

(iv) $(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1, \otimes_{i=1}^3 p_i^* \mathcal{O}_{\mathbb{P}^1}(1))$, where p_i is the ith projection.

(v) A hyperquadric fibration over a smooth curve.

(vi) $(\mathbb{P}_S(\mathcal{E}), H(\mathcal{E}))$, where S is a smooth projective surface and $\mathcal{E}$ is an ample and spanned vector bundle on S with $c_2(\mathcal{E}) = 2$.

(vii) A reduction (M, A) of (X, L) is a Veronese fibration over a smooth curve C, that is, M is a $\mathbb{P}^2$-bundle over C and $A|_F = \mathcal{O}_{\mathbb{P}^2}(2)$ for every fiber F of it.

(A) If (X, L) is one of the types (i) (ii), (iii) and (iv), then we see from [7, Example 3.1] that $b_2(X, L) = h^2(X, \mathbb{C}) + 1$.

(B) Next we consider the case of (vi). In this case, since $\mathcal{E}$ is an ample and spanned vector bundle with $c_2(\mathcal{E}) = 2$, we see from [10] and [12, Theorem 6.1] that $(S, \mathcal{E})$ is one of the types from (e.1) to (e.4) in Theorem 3.1.

(C) Next we consider the case of (v) and we use notation in Notation 2.2.1 (1). Here we note that $h^2(X, \mathbb{C}) = 2$ in this case. Since $b_2(X, L) = h^2(X, \mathbb{C}) + 1$ and $h^2(X, \mathbb{C}) = 2$, we see from [7, Example 3.1 (5)] that $2e + 3b = 1$. On the other hand, from the fact that $L^n = 2e + b > 0$ and $2e + (n + 1)b \geq 0$ by [3, (3.3)], we get the following.

Claim 3.1 *$n = 3$, $e = 2$ and $b = -1$.*

Proof. If $b > 0$, then $2e + 3b = 2e + b + 2b \geq 3$ and this is impossible. So we have $b \leq 0$. If $b = 0$, then $2e = 1$ and this is also impossible. Therefore we get $b < 0$. Then $1 = 2e + 3b = 2e + (n + 1)b - (n - 2)b \geq -(n - 2)b \geq (n - 2)$. So we have $n = 3$ because we assume $n \geq 3$. Since $1 + b = 2e + 4b = 2e + (n + 1)b \geq 0$, we have $b = -1$. Hence $e = 2$ because $2e + 3b = 1$. □

Hence $L^3 = 2e + b = 3$. Since L is ample and spanned, we have $h^0(L) \geq n + 1 = 4$. Therefore $\Delta(X, L) = 3 + L^3 - h^0(L) \leq 2$. Now since $g(X, L) > 0$, we see that $\Delta(X, L) \geq 1$ holds by [4, (12.1) Theorem].

(C.1) If $\Delta(X, L) = 1$, then by a result of Fujita [4, (6.7) Corollary] we see that (X, L) is a hypercubic in $\mathbb{P}^4$. But then $\text{Pic}(X) \cong \text{Pic}(\mathbb{P}^4)$ by the Lefschetzs theorem. Hence $\text{Pic}(X) \cong \mathbb{Z}$. But this is impossible because X is a hyperquadric fibration over a smooth curve.

(C.2) If $\Delta(X, L) = 2$, then there exists a triple covering $\pi : X \to \mathbb{P}^3$ such that $L = \pi^*(\mathcal{O}_{\mathbb{P}^3}(1))$. By a Barth's theorem (see [11, Theorem 7.1.15]) we have $H^1(\mathbb{P}^3, \mathbb{C}) \cong H^1(X, \mathbb{C})$. In particular we have $q(X) = 0$. Hence $g(C) = 0$. Let $\mathcal{E} := f_*(L)$. Then $\mathcal{E}$ is decomposable and we set $\mathcal{E} := \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1}(a_2) \oplus \mathcal{O}_{\mathbb{P}^1}(a_3) \oplus \mathcal{O}_{\mathbb{P}^1}(a_4)$. But we can easily see that this is impossible because $h^0(\mathcal{E}) = h^0(L) = 4$ and $a_1 + a_2 + a_3 + a_4 = e = 2$.

(D) Finally we consider the case (vii). We use notation in Notation 2.2.1 (2). From [7, Example 3.1 (7.4)] we have

$$2e + 3b = 1 \tag{1}$$

because $b_2(X, L) = h^2(X, \mathbb{C}) + 2e + 3b$. Here we note that by [7, Remark 2.6]

$$g_1(M, A) = 2e + 2b + 1. \tag{2}$$

Here we note that $g_1(M, A) \geq 2$ in this case because $K_M + 2A$ is ample. Hence by (2) we have

$$e + b \geq 1. \tag{3}$$

We also note that by [7, Remark 2.6]

$$e + 2b + 2g(C) - 2 = 0. \tag{4}$$

Hence we see from (1) and (4)

$$b = 3 - 4g(C), \tag{5}$$
$$e = 6g(C) - 4. \tag{6}$$

By (3), (5) and (6), we get $2g(C) - 1 = b + e \geq 1$, that is, $g(C) \geq 1$.

Claim 3.2 *(X, L) is isomorphic to (M, A).*

Proof. Assume that (X, L) is not isomorphic to (M, A). Then we have $L^3 \leq 3$ because $A^3 = 8e + 12b = 4$.
(I) If $L^3 \leq 2$, then $\Delta(X, L) \leq 1$ and we see from [4, (5.10) Theorem and (6.13) Corollary] that $q(X) = 0$. But this is impossible because $q(X) = g(C) \geq 1$ in this case.
(II) If $L^3 = 3$, then we have $\Delta(X, L) \leq 2$.
(II.1) If $\Delta(X, L) \leq 1$, then by [4, (5.10) Theorem and (6.7) Corollary] we have $q(X) = 0$ and this is impossible.
(II.2) If $\Delta(X, L) = 2$, then by the same argument as (C.2) above we can prove that $q(X) = 0$ and this is also impossible.
Therefore we get the assertion of Claim 3.2. □

So we have $L^3 = 4$ and $\Delta(X, L) \leq 3$. Since $g_1(X, L) \geq 2$, we have $\Delta(X, L) \geq 1$ by [4, (4.2) Theorem and (5.10) Theorem].

(D.1) If $\Delta(X, L) = 1$, then we see from [4, (6.8) Corollary] that X is a complete intersection of $(2, 2)$ in $\mathbb{P}^5$. Hence $q(X) = 0$ and this is impossible.

(D.2) If $\Delta(X, L) = 2$, then $h^0(L) = 5$. Since L is spanned, by using the morphism given by $\Gamma(L)$ we see that there exist a projective variety W of dimension 3 and a surjective morphism $\rho : X \to W \subset \mathbb{P}^4$ such that one of the following holds:

(D.2.1) $\deg \rho = 1$ and $\deg W = 4$.
(D.2.2) $\deg \rho = 2$ and $\deg W = 2$.

(D.2.1) First we consider the case (D.2.1) above. Then by [4, (10.8.1) in Chapter I] we see that $g_1(X, L) \leq 3$. We also note that $g_1(X, L) \geq 2$ in the case (vii).

(D.2.1.1) If $g_1(X, L) = 3$, then we can prove that W is normal by [4, (10.8.1) in Chapter I]. Hence by Zariski's Main Theorem we infer that X is isomorphic to W and is a hypersurface of degree 4 in $\mathbb{P}^4$. In particular $q(X) = 0$ and this is impossible.

(D.2.1.2) If $g_1(X, L) = 2$, then we have $2e + 2b + 1 = 2$, that is, $2e + 2b = 1$. But this is also impossible.

(D.2.2) Next we consider the case (D.2.2). Then W becomes smooth if $n \geq 3$ by [4, (10.8.2) in Chapter I]. We note that $\mathrm{Pic}(W) \cong \mathbb{Z}$ because $\mathrm{Pic}(W) \cong \mathrm{Pic}(\mathbb{P}^4)$ by the Lefschetz theorem. Let $\mathcal{O}_W(1)$ be the ample generator of $\mathrm{Pic}(W)$. Then $K_W = \mathcal{O}_W(-3)$ and $L = \rho^*(\mathcal{O}_W(1))$. Let B be the branch locus of ρ. Then $B \in |\mathcal{O}_W(2b)|$ for some integer b, and $K_X = \rho^*(\mathcal{O}_W(b-3))$ (see [4, (6.11) and (6.12)]). But since the nef value of L is equal to $3/2$, the equality $b - 3 + (3/2) \cdot 1 = 0$ must hold. But this is impossible because b is an integer.

Therefore we get the assertion. □

Remark 3.1 In the type (c) of Theorem 3.1, L is very ample.
Proof. In this case (X, L) is a Del Pezzo manifold. Hence $g_1(X, L) = 1$ and $\Delta(X, L) = 1$. Moreover since X is smooth, we see that L has a ladder (see [4, (6.1.3) and (6.1.4)]). Since $L^3 = 7$, we have $L^3 > 2\Delta(X, L)$. Therefore these enable us to prove that L is very ample by using [4, (3.5) Theorem 3)]. □

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