A note on papers Math. Z. 250 (2005), 573–597 and Int. J. Math. 17 (2006), 761–789.

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In our papers [3] and [4] we gave the definition of the scroll (resp. quadric fibration) as follows:

Definition 1 (See [3, Definition 1.3] and [4, Definition 1.3].) Let (X, L) be a polarized manifold of dimension n. We say that (X, L) is a *scroll* (resp. *quadric fibration*) *over a normal projective variety* Y *with* $\dim Y = m$ if there exists a surjective morphism with connected fibers $f : X \to Y$ such that $K_X + (n - m + 1)L = f^*A$ (resp. $K_X + (n - m)L = f^*A$) for some ample line bundle A on Y.

This is the adjunction theoretic definition. But in some results in [3] and [4] we used other definition of the scroll (resp. quadric fibration). So here we would like to give more precise statements about these. First we note the following:

Remark 1 (1) If (X, L) is a scroll over a smooth curve C (resp. a smooth projective surface S) with $\dim X = n \geq 3$, then by [2, (3.2.1) Theorem] and [1, Proposition 3.2.1] there exists an ample vector bundle $\mathcal{E}$ of rank n (resp. $n-1$) on C (resp. S) such that $(X, L) \cong (\mathbb{P}_C(\mathcal{E}), H(\mathcal{E}))$ (resp. $(\mathbb{P}_S(\mathcal{E}), H(\mathcal{E}))$). Here $\mathbb{P}_Y(\mathcal{E})$ denotes the projective space bundle over a smooth projective variety Y and $H(\mathcal{E})$ denotes the tautological line bundle.

(2) In some papers, (X, L) is called a (classical) scroll over a smooth projective variety Y if $(X, L) \cong (\mathbb{P}_Y(\mathcal{E}), H(\mathcal{E}))$, where $\mathcal{E}$ is a vector bundle on Y.

Remark 2 Assume that (X, L) is a quadric fibration over a smooth curve C with $\dim X = n \geq 3$. Let $f : X \to C$ be its morphism. By [2, (3.2.6) Theorem] and the proof of [5, Lemma (c) in Section 1], we see that (X, L) is one of the following:

(a) f is the contraction morphism of an extremal ray, and every fiber of f is irreducible and reduced. We put $\mathcal{E} := f_*(L)$. Then $\mathcal{E}$ is a locally free sheaf of rank $n + 1$ on C. Let $\pi : \mathbb{P}_C(\mathcal{E}) \to C$ be the projection. Then there exists an embedding $i : X \hookrightarrow \mathbb{P}_C(\mathcal{E})$ such that $f = \pi \circ i$, $X \in |2H(\mathcal{E}) + \pi^*(B)|$ for some

$B \in \mathrm{Pic}(C)$ and $L = H(\mathcal{E})|_X$. Moreover $\rho(X) = \rho(C) + 1 = 2$. Therefore $h^2(X, \mathbb{C}) = 2$ in this case.

(b) X is a $\mathbb{P}^1$-bundle over a smooth surface and $L|_F = \mathcal{O}_{\mathbb{P}^1}(1)$ for every fiber F.

So if (X, L) is not the type (a) in Remark 2, then we may assume that there exists an ample vector bundle $\mathcal{F}$ of rank 2 on a smooth projective surface S such that $(X, L) \cong (\mathbb{P}_S(\mathcal{F}), H(\mathcal{F}))$. In particular $\dim X = 3$ in this case.

Definition 2 (X, L) is called a *hyperquadric fibration over a smooth curve C* if (X, L) is a quadric fibration over C which is the type (a) in Remark 2.

By these remarks, we can rephrase [3, Theorem 1.2] and [4, Theorem 1.2] as follows: (The types (5) and (6) are slightly changed.)

Theorem 1 *Let (X, L) be a polarized manifold with $n = \dim X \geq 3$. Then (X, L) is one of the following types.*

(1) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$.

(2) $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1))$.

(3) *A scroll over a smooth projective curve.*

(4) $K_X \sim -(n-1)L$, *that is, (X, L) is a Del Pezzo manifold.*

(5) *A hyperquadric fibration over a smooth curve.*

(6) $(\mathbb{P}_S(\mathcal{E}), H(\mathcal{E}))$, *where S is a smooth projective surface and $\mathcal{E}$ is an ample vector bundle of rank $n-1$ on S.*

(7) *Let (M, A) be a reduction of (X, L).*

(7.1) $n = 4$, $(M, A) = (\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2))$.

(7.2) $n = 3$, $(M, A) = (\mathbb{Q}^3, \mathcal{O}_{\mathbb{Q}^3}(2))$.

(7.3) $n = 3$, $(M, A) = (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3))$.

(7.4) $n = 3$, *M is a $\mathbb{P}^2$-bundle over a smooth curve C and for any fiber F' of it, $(F', A|_{F'}) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$.*

(7.5) $K_M \sim -(n-2)A$, *that is, (M, A) is a Mukai manifold.*

(7.6) *(M, A) is a Del Pezzo fibration over a smooth curve.*

(7.7) *(M, A) is a quadric fibration over a normal surface.*

(7.8) *$n \geq 4$, and there exist a normal projective variety W with $\dim W = 3$ and a fiber space $\Phi : M \to W$ such that for a general fiber F' of Φ, $(F', A|_{F'}) \cong (\mathbb{P}^{n-3}, \mathcal{O}_{\mathbb{P}^{n-3}}(1))$.*

(7.9) *$K_M + (n-2)A$ is nef and big.*

Now we will modify some statements in [3] and [4].
(1.0) A statement in [3, Notation 1.3] should be modified as follows: "Assume that (X, L) is a hyperquadric fibration over a smooth curve C."
(1.1) The statement in [3, Theorem 3.1.1 (2)] should be modified as follows:
"$8\chi_2^H(X, L) \geq (K_X + (n-2)L)^2 L^{n-2}$ holds if (X, L) is neither $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ nor $(\mathbb{P}_S(\mathcal{E}), H(\mathcal{E}))$, where S is a smooth projective surface and $\mathcal{E}$ is an ample vector bundle of rank $n-1$ on S."
(1.2) The statement in [3, Theorem 3.1.1 (4)] should be modified as follows:
"If $(X, L) \cong (\mathbb{P}_S(\mathcal{E}), H(\mathcal{E}))$, where S is a smooth projective surface and $\mathcal{E}$ is an ample vector bundle of rank $n-1$ on S, then

$$(K_X + (n-2)L)^2 L^{n-2} \begin{cases} \leq 8\chi_2^H(X, L) & \text{if } \kappa(S) \neq 2, \\ < 9\chi_2^H(X, L) & \text{if } \kappa(S) = 2.\text{"} \end{cases}$$

Here we also note the following:

Claim 1 *If (X, L) is a quadric fibration over a smooth curve C, then $(K_X + (n-2)L)^2 L^{n-2} \leq 8\chi_2^H(X, L)$ holds.*

Proof. If (X, L) is the type (a) in Remark 2, then this inequality holds by the proof of [3, Theorem 3.1.1].

Assume that (X, L) is the type (b) in Remark 2. Then X is a $\mathbb{P}^1$-bundle over a smooth surface S. Let $f : X \to C$ be the quadric fibration over C and let $\pi : X \to S$ be the projection. Then there exists a surjective morphism $\delta : S \to C$ such that $f = \delta \circ \pi$. Since $q(F_f) = 0$ for any general fiber F_f of f, we have $q(F_\delta) = 0$ for any general fiber F_δ of δ. Therefore $\kappa(S) = -\infty$ because $\dim F_\delta = 1$. So by (1.2) above, we have $(K_X + (n-2)L)^2 L^{n-2} \leq 8\chi_2^H(X, L)$. Therefore we get the assertion. □

(1.3) At the second line of the case (5) in the proof of [3, Theorem 3.1.1] we should add the following statement after "$\cdots$ and $\chi_2^H(X, L) = 1 - q(X) = 1 - g(C)$." :
"Assume that (X, L) is a hyperquadric fibration over C, that is, (X, L) satisfies the type (a) in Remark 2."
(1.4) The statement of the case (6) in the proof of [3, Theorem 3.1.1] should be modified as follows:
"The case where $(X, L) \cong (\mathbb{P}_S(\mathcal{E}), H(\mathcal{E}))$, where S is a smooth projective surface and $\mathcal{E}$ is an ample vector bundle of rank $n-1$ on S."
(1.5) Here we note that the example which was given in [3, Example 3.1.1] is a scroll over a smooth surface in the sense of Definition 1.

(2.0) A statement in [4, Notation 1.2] should be modified as follows: "Assume that (X, L) is a hyperquadric fibration over a smooth curve C."
(2.1) The statement in [4, Theorem 2.2 (B)] should be modified as follows:
"(X, L) is a hyperquadric fibration over an elliptic curve C, and one of the following holds."

(2.2) The statement in [4, Theorem 2.2 (C)] should be modified as follows:
"$(X, L) \cong (\mathbb{P}_S(\mathcal{E}), H(\mathcal{E}))$, where S is a smooth projective surface and $\mathcal{E}$ is an ample vector bundle of rank $n-1$ on S, and $(S, \mathcal{E})$ is one of the following."
(2.3) The statement of the cases (b) and (c) in the proof of [4, Theorem 2.2] should be modified as follows:

(b) A hyperquadric fibration over a smooth curve.

(c) $(X, L) \cong (\mathbb{P}_S(\mathcal{E}), H(\mathcal{E}))$, where S is a smooth projective surface and $\mathcal{E}$ is an ample vector bundle of rank $n-1$ on S.

(2.4) The statement in [4, the proof of Theorem 2.2, P774 line $\uparrow$ 3] should be modified as follows:
"(b) The case in which (X, L) is a hyperquadric fibration over a smooth curve."
(2.5) The statement in [4, the proof of Theorem 2.2, P775 line $\uparrow$ 5] should be modified as follows:
"(c) The case where $(X, L) \cong (\mathbb{P}_S(\mathcal{E}), H(\mathcal{E}))$, where S is a smooth projective surface and $\mathcal{E}$ is an ample vector bundle of rank $n-1$ on S."
(2.6) The statement in [4, Theorem 2.4 (B)] should be modified as follows:
"(X, L) is a hyperquadric fibration over an elliptic curve C, and one of the following holds."
(2.7) The statement in [4, Theorem 2.4 (C)] should be modified as follows:
"$(X, L) \cong (\mathbb{P}_S(\mathcal{E}), H(\mathcal{E}))$, where S is a smooth projective surface and $\mathcal{E}$ is an ample vector bundle of rank 2 on S, and $(S, \mathcal{E})$ is one of the following."

(3) Here we would like to fix an error of the statement in [4, Problem 3.2.1].

Table of an error.

Page	Line	Error	Correct
786	19	$m \geq 1$	$m \geq 2$

Actually, to be pricise, Tsuji gave this problem for $m \geq 2$. But it is also important to consider the case where $m = 1$.

References

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