Calculations of sectional classes of special polarized manifolds

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In this note, we will calculate the ith sectional class $\mathrm{cl}_i(X, L)$ of some special polarized manifolds (X, L).

1 Preliminaties

Here we are going to calculate the ith sectional class $\mathrm{cl}_i(X, L)$ of some special polarized manifolds (X, L) with $n = \dim X \geq 3$ by using its ith sectional Euler number $e_i(X, L)$ (see [4, Definition 3.1 (1)]).

Definition 1.1 Let (X, L) be a polarized manifold of dimension n. Then for every integer i with $0 \leq i \leq n$ the ith sectional class of (X, L) is defined by the following.

$$\mathrm{cl}_i(X, L) := \begin{cases} e_0(X, L), & \text{if } i = 0, \\ (-1)\{e_1(X, L) - 2e_0(X, L)\}, & \text{if } i = 1, \\ (-1)^i\{e_i(X, L) - 2e_{i-1}(X, L) + e_{i-2}(X, L)\}, & \text{if } 2 \leq i \leq n. \end{cases}$$

Definition 1.2 Let (X, L) be a polarized manifold of dimension n.

(i) The *deficiency* of (X, L) is defined by the following.

$$\mathrm{def}(X, L) := \min\{\ i \mid 0 \leq i \leq n,\ \mathrm{cl}_{n-i}(X, L) \neq 0\}$$

(ii) The *codegree* of (X, L) is defined by the following.

$$\mathrm{codeg}(X, L) := \mathrm{cl}_{n-k}(X, L),$$

where $k = \mathrm{def}(X, L)$.

Notation 1.1 (1) Let (X, L) be a hyperquadric fibration over a smooth curve C. We put $\mathcal{E} := f_*(L)$. Then $\mathcal{E}$ is a locally free sheaf of rank $n + 1$ on C. Let $\pi : \mathbb{P}_C(\mathcal{E}) \to C$ be the projective bundle. Then $X \in |2H(\mathcal{E}) + \pi^*(B)|$ for some $B \in \mathrm{Pic}(C)$ and $L = H(\mathcal{E})|_X$, where $H(\mathcal{E})$ is the tautological line bundle of $\mathbb{P}_C(\mathcal{E})$. We put $e := \deg \mathcal{E}$ and $b := \deg B$.

(2) (See [2, (13.10) Chapter II].) Let (M, A) be a $\mathbb{P}^2$-bundle over a smooth curve C and $A|_F = \mathcal{O}_{\mathbb{P}^2}(2)$ for any fiber F of it. Let $f : M \to C$ be the fibration and $\mathcal{E} := f_*(K_M + 2A)$. Then $\mathcal{E}$ is a locally free sheaf of rank 3 on C, and $M \cong \mathbb{P}_C(\mathcal{E})$ such that $H(\mathcal{E}) = K_M + 2A$. In this case, $A = 2H(\mathcal{E}) + f^*(B)$ for a line bundle B on C, and by the canonical bundle formula $K_M = -3H(\mathcal{E}) + f^*(K_C + \det\mathcal{E})$. Here we set $e := \deg \mathcal{E}$ and $b := \deg B$.

Definition 1.3 Let $\mathcal{F}$ be a vector bundle on a smooth projective variety X. Then for every integer j with $j \geq 0$, the *jth Segre class $s_j(\mathcal{F})$ of $\mathcal{F}$* is defined by the following equation: $c_t(\mathcal{F}^\vee)s_t(\mathcal{F}) = 1$, where $\mathcal{F}^\vee := Hom_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$, $c_t(\mathcal{F}^\vee)$ is the Chern polynomial of $\mathcal{F}^\vee$ and $s_t(\mathcal{F}) = \sum_{j\geq 0} s_j(\mathcal{F})t^j$.

Remark 1.1 (a) Let $\mathcal{F}$ be a vector bundle on a smooth projective variety X. Let $\tilde{s}_j(\mathcal{F})$ be the jth Segre class which is defined in [7, Chapter 3]. Then $s_j(\mathcal{F}) = \tilde{s}_j(\mathcal{F}^\vee)$.
(b) For every integer i with $1 \leq i$, $s_i(\mathcal{F})$ can be written by using the Chern classes $c_j(\mathcal{F})$ with $1 \leq j \leq i$. (For example, $s_1(\mathcal{F}) = c_1(\mathcal{F})$, $s_2(\mathcal{F}) = c_1(\mathcal{F})^2 - c_2(\mathcal{F})$, and so on.)

2 Calculations

Example 2.1 (i) The case where (X, L) is $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$.
Then by [6, Example 3.1] we have

$$\mathrm{cl}_i(X, L) = \begin{cases} 1, & \text{if } i = 0, \\ 0, & \text{if } i \geq 1. \end{cases}$$

(ii) The case where (X, L) is $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1))$.
Then by [6, Example 3.2] we have $\mathrm{cl}_i(X, L) = 2$ for $0 \leq i \leq n$. In this case, $\mathrm{def}(X, L) = 0$ and $\mathrm{codeg}(X, L) = 2$.
(iii) The case where (X, L) is $(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2))$.
Then by [6, Example 3.3] we have

$$\mathrm{cl}_i(X, L) = \begin{cases} 16, & \text{if } i = 0, \\ 40, & \text{if } i = 1, \\ 40, & \text{if } i = 2, \\ 20, & \text{if } i = 3, \\ 5, & \text{if } i = 4. \end{cases}$$

In this case, $\mathrm{def}(X, L) = 0$ and $\mathrm{codeg}(X, L) = 5$.
(iv) The case where (X, L) is $(\mathbb{Q}^3, \mathcal{O}_{\mathbb{Q}^3}(2))$.
Then by [6, Example 3.4] we have

$$\mathrm{cl}_i(X, L) = \begin{cases} 16, & \text{if } i = 0, \\ 40, & \text{if } i = 1, \\ 40, & \text{if } i = 2, \\ 20, & \text{if } i = 3. \end{cases}$$

In this case, $\mathrm{def}(X, L) = 0$ and $\mathrm{codeg}(X, L) = 20$.
(v) The case where (X, L) is $(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3))$.
Then by [6, Example 3.5] we have

$$\mathrm{cl}_i(X, L) = \begin{cases} 27, & \text{if } i = 0, \\ 72, & \text{if } i = 1, \\ 72, & \text{if } i = 2, \\ 32, & \text{if } i = 3. \end{cases}$$

In this case, $\mathrm{def}(X, L) = 0$ and $\mathrm{codeg}(X, L) = 32$.
(vi) The case where (X, L) is a Veronese fibration over a smooth curve C.
Here we use Notation 1.1 (2). Then by [6, Example 3.6] we have

$$\mathrm{cl}_i(X, L) = \begin{cases} 8e + 12b, & \text{if } i = 0, \\ 20e + 28b, & \text{if } i = 1, \\ 36e + 47b, & \text{if } i = 2, \\ 41e + 52b, & \text{if } i = 3. \end{cases}$$

First we note that

$$8e + 12b = L^3 \tag{1}$$
$$2g(C) - 2 + e + 2b = 0 \tag{2}$$
$$g(X, L) = 1 + 2e + 2b \tag{3}$$

Here we set $L^3 = 4m$. Then m is an integer with $m \geq 1$. We see from (1) and (2) that $b = 4(1 - g(C)) - m$ and $e = 6(g(C) - 1) + 2m$. Therefore

$$\mathrm{cl}_1(X, L) = 20e + 28b = 12m + 8(g(C) - 1) > 0.$$

Next we consider $\mathrm{cl}_2(X, L)$. Then

$$\mathrm{cl}_2(X, L) = 36e + 47b = 25m + 28(g(C) - 1).$$

If $g(C) = 0$ and $m = 1$, then we have $e = -4$ and $b = 3$. But then by (3) we have $g(X, L) = -1 < 0$ and this is impossible. Hence $g(C) \geq 1$ or $m \geq 2$, and we get

$$\mathrm{cl}_2(X, L) \geq 25m + 28(g(C) - 1) \geq 22.$$

Finally we consider $\mathrm{cl}_3(X, L)$. Then

$$\mathrm{cl}_3(X, L) = 41e + 52b = 30m + 38(g(C) - 1).$$

By the same argument as above, the case where $g(C) = 0$ and $m = 1$ does not occur. Hence $g(C) \geq 1$ or $m \geq 2$, and we get

$$\mathrm{cl}_3(X, L) \geq 30m + 38(g(C) - 1) \geq 22.$$

Therefore $\mathrm{def}(X, L) = 0$ and $\mathrm{codeg}(X, L) = 30m + 38(g(C) - 1)$.
(vii) The case where (X, L) is a Del Pezzo manifold with $n = \dim X \geq 3$.
Here we note that by [2, (8.11) Theorem], we have $L^n \leq 8$ and (X, L) is one of the following:

(vii.1) $(X, L) \cong (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2))$.
Then by [6, Example 3.7 (3.7.1)] we have

$$\mathrm{cl}_i(X, L) = \begin{cases} 8, & \text{if } i = 0, \\ 16, & \text{if } i = 1, \\ 12, & \text{if } i = 2, \\ 4, & \text{if } i = 3. \end{cases}$$

In this case, $\mathrm{def}(X, L) = 0$ and $\mathrm{codeg}(X, L) = 4$.

(vii.2) X is the blowing up of $\mathbb{P}^3$ at a point and $L = \pi^*(\mathcal{O}_{\mathbb{P}^3}(2)) - E$, where $\pi : X \to \mathbb{P}^3$ is its birational morphism and E is the exceptional divisor. Then by [6, Example 3.7 (3.7.2)] we have

$$\mathrm{cl}_i(X, L) = \begin{cases} 7, & \text{if } i = 0, \\ 14, & \text{if } i = 1, \\ 12, & \text{if } i = 2, \\ 4, & \text{if } i = 3. \end{cases}$$

In this case, $\mathrm{def}(X, L) = 0$ and $\mathrm{codeg}(X, L) = 4$.

(vii.3) (X, L) is either

$$(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1, \otimes_{i=1}^3 p_i^* \mathcal{O}_{\mathbb{P}^1}(1)), (\mathbb{P}^2 \times \mathbb{P}^2, \otimes_{i=1}^2 p_i^* \mathcal{O}_{\mathbb{P}^2}(1)) \text{ or } (\mathbb{P}_{\mathbb{P}^2}(T_{\mathbb{P}^2}), H(T_{\mathbb{P}^2}))$$

where p_i is the ith projection and $T_{\mathbb{P}^2}$ is the tangent bundle of $\mathbb{P}^2$.

(vii.3.1) The case where $(X, L) \cong (\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1, \otimes_{i=1}^3 p_i^* \mathcal{O}_{\mathbb{P}^1}(1))$.
Then by [6, Example 3.7 (3.7.3.1)] we have

$$\mathrm{cl}_i(X, L) = \begin{cases} 6, & \text{if } i = 0, \\ 12, & \text{if } i = 1, \\ 12, & \text{if } i = 2, \\ 4, & \text{if } i = 3. \end{cases}$$

In this case, $\mathrm{def}(X, L) = 0$ and $\mathrm{codeg}(X, L) = 4$.

(vii.3.2) The case where $(X, L) \cong (\mathbb{P}^2 \times \mathbb{P}^2, \otimes_{i=1}^2 p_i^* \mathcal{O}_{\mathbb{P}^2}(1))$.
Then by [6, Example 3.7 (3.7.3.2)] we have

$$\mathrm{cl}_i(X, L) = \begin{cases} 6, & \text{if } i = 0, \\ 12, & \text{if } i = 1, \\ 12, & \text{if } i = 2, \\ 6, & \text{if } i = 3, \\ 3, & \text{if } i = 4. \end{cases}$$

In this case, $\mathrm{def}(X, L) = 0$ and $\mathrm{codeg}(X, L) = 3$.

(vii.3.3) The case where $(X, L) \cong (\mathbb{P}_{\mathbb{P}^2}(T_{\mathbb{P}^2}), H(T_{\mathbb{P}^2}))$.
Then by [6, Example 3.7 (3.7.3.3)] we have

$$\mathrm{cl}_i(X, L) = \begin{cases} 6, & \text{if } i = 0, \\ 12, & \text{if } i = 1, \\ 12, & \text{if } i = 2, \\ 6, & \text{if } i = 3. \end{cases}$$

In this case, $\mathrm{def}(X, L) = 0$ and $\mathrm{codeg}(X, L) = 6$.

(vii.4) The case where (X, L) is a linear section of the Grassmann variety $\mathrm{Gr}(5, 2)$ parametrizing lines in $\mathbb{P}^4$, embedded in $\mathbb{P}^9$ via the Plücker embedding. Then $3 \leq n \leq 6$ and $L^n = 5$.

By [6, Example 3.7 (3.7.4)] we have

$$\mathrm{cl}_i(X, L) = \begin{cases} 5, & \text{if } i = 0, \\ 10, & \text{if } i = 1, \\ 12, & \text{if } i = 2, \\ 10, & \text{if } i = 3, \\ 5, & \text{if } i = 4 \text{ and } 4 \leq n \leq 6, \\ 0, & \text{if } i = 5 \text{ and } 5 \leq n \leq 6, \\ 0, & \text{if } i = 6 \text{ and } n = 6. \end{cases}$$

In this case, if $n = 6$ (resp. 5, 4, 3), then $\mathrm{def}(X, L) = 2$ (resp. 1, 0, 0) and $\mathrm{codeg}(X, L) = 5$ (resp. 5, 5, 10).

(vii.5) The case where (X, L) is a complete intersection of two hyperquadrics in $\mathbb{P}^{n+2}$.

Then by [6, Example 3.7 (3.7.5)] we have

$$\mathrm{cl}_i(X, L) = 4i + 4.$$

In this case, $\mathrm{def}(X, L) = 0$ and $\mathrm{codeg}(X, L) = 4n + 4$.

(vii.6) The case where X is a hypercubic in $\mathbb{P}^{n+1}$ and $L = \mathcal{O}_X(1)$.

Then by [6, Example 3.7 (3.7.6)] we have

$$\mathrm{cl}_i(X, L) = 3 \cdot 2^i.$$

In this case, $\mathrm{def}(X, L) = 0$ and $\mathrm{codeg}(X, L) = 3 \cdot 2^n$.

In general, the following holds by Definitions 1.1, 1.2 and [6, Lemma 3.3] (see also [8, (9) Proposition in II]).

Proposition 2.1 *If X is a hypersurface of degree m in $\mathbb{P}^{n+1}$, then*

$$\mathrm{cl}_i(X, L) = m(m-1)^i, \ \mathrm{def}(X, L) = 0 \textit{ and } \mathrm{codeg}(X, L) = m(m-1)^n.$$

(vii.7) The case where X is a double covering of $\mathbb{P}^n$ branched along a smooth hypersurface of degree 4, and L is the pull-back of $\mathcal{O}_{\mathbb{P}^n}(1)$.

Then by [6, Example 3.7 (3.7.7)] we have

$$\mathrm{cl}_i(X, L) = \begin{cases} 2, & \text{if } i = 0, \\ 4 \cdot 3^{i-1}, & \text{if } i \geq 1. \end{cases}$$

In this case, $\mathrm{def}(X, L) = 0$ and $\mathrm{codeg}(X, L) = 4 \cdot 3^{n-1}$.

In general, we can prove the following by using [6, Lemma 3.4].

Proposition 2.2 *If X is a double covering of $\mathbb{P}^n$ branched along a smooth hypersurface of degree m, and L is the pull back of $\mathcal{O}_{\mathbb{P}^n}(1)$, then for $i \geq 1$ we have*

$$\mathrm{cl}_i(X, L) = m(m-1)^{i-1}, \ \mathrm{def}(X, L) = 0 \textit{ and } \mathrm{codeg}(X, L) = m(m-1)^{n-1}.$$

(vii.8) The case where (X, L) is a weighted hypersurface of degree 6 in the weighted projective space $\mathbb{P}(3, 2, 1, \ldots, 1)$.

Then by [6, Example 3.7 (3.7.8)] we have

$$\mathrm{cl}_i(X, L) = \begin{cases} 1, & \text{if } i = 0, \\ 2, & \text{if } i = 1, \\ 12 \cdot 5^{i-2}, & \text{if } i \geq 2. \end{cases}$$

In this case, $\mathrm{def}(X, L) = 0$ and $\mathrm{codeg}(X, L) = 12 \cdot 5^{n-2}$.

(viii) The case where (X, L) is a hyperquadric fibration over a smooth curve C.
Here we use notation in Notation 1.1 (1). Then by [6, Example 3.8] we have

$$\mathrm{cl}_i(X, L) = \begin{cases} 2e + b, & \text{if } i = 0, \\ 6e + 4b + 4(g(C) - 1), & \text{if } i = 1, \\ 8e + 4ib + 4(g(C) - 1), & \text{if } i \geq 2. \end{cases}$$

Here we consider a lower bound of $\mathrm{cl}_i(X, L)$ for $i \geq 1$.

Proposition 2.3 *Let (X, L) be a hyperquadric fibration over a smooth curve C. If $i \geq 1$, then* $\mathrm{cl}_i(X, L) \geq 4$.

Proof. Then we use the following inequalities.

$$\begin{aligned} 2e+b &> 0 && (4) \\ 2e+(n+1)b &\geq 0 && (5) \end{aligned}$$

(A) First we consider the case $i=1$. Then $g(X,L)\geq 2$ holds because (X,L) is a hyperquadric fibration over a smooth curve. Hence by definition we have $\mathrm{cl}_1(X,L)=2(g(X,L)+L^n-1)\geq 4$.
(B) Next we consider the case $i\geq 2$.
(B.1) If $b<0$, then by (5) we have

$$\begin{aligned} 2e+ib &= 2e+(n+1)b-(n+1-i)b && (6) \\ &\geq -(n+1-i)b \\ &\geq n+1-i. \end{aligned}$$

Hence

$$\begin{aligned} \mathrm{cl}_i(X,L) &= 8e+4ib+4(g(C)-1) \\ &= 4(2e+ib)+4(g(C)-1) \\ &\geq 4(n+1-i)+4(g(C)-1) \\ &= 4(n-i)+4g(C) \\ &\geq 0. \end{aligned}$$

If $\mathrm{cl}_i(X,L)=0$, then $i=n$ and $g(C)=0$. Then by (5) we have $0=\mathrm{cl}_i(X,L)=4(2e+(n+1)b)-4b-4\geq -4b-4\geq 0$ and we get $2e+(n+1)b=0$ and $b=-1$. Since $g(C)=0$, we see that $\mathcal{E}$ can be expressed as

$$\mathcal{E}=\bigoplus_{i=0}^{n}\mathcal{O}(e_i).$$

We may assume that $e_0\leq\cdots\leq e_n$. Since $b=-1$, we see that $e_0\geq 1$ by the same argument as in the proof of [1, Lemma (3.19)]. Hence

$$e=\sum_{i=0}^{n}e_i\geq n+1.$$

But this is impossible because

$$e=-\frac{(n+1)}{2}b=\frac{(n+1)}{2}.$$

Hence $\mathrm{cl}_i(X,L)>0$ in this case.
(B.2) If $b\geq 0$, then by (4) we have $2e+ib=2e+b+(i-1)b\geq 1+(i-1)b$. Hence

$$\begin{aligned} \mathrm{cl}_i(X,L) &= 8e+4ib+4(g(C)-1) \\ &\geq 4(i-1)b+4g(C) \\ &\geq 0. \end{aligned}$$

If $\mathrm{cl}_i(X,L)=0$, then $b=0$ and $g(C)=0$. Then we have $\mathrm{cl}_i(X,L)=8e-4$. But since $\mathrm{cl}_i(X,L)=0$, we have $e=\frac{1}{2}$ and this is impossible. Therefore $\mathrm{cl}_i(X,L)>0$ holds in this case, too.

Since $\mathrm{cl}_i(X,L)$ for $i\geq 2$ is divided by 4, we see that $\mathrm{cl}_i(X,L)\geq 4$. □

Hence we see from Proposition 2.3 that $\mathrm{def}(X,L)=0$ and $\mathrm{codeg}(X,L)=8e+4nb+4(g(C)-1)$.

(ix) The case where (X, L) is a scroll over a smooth curve C with $n = \dim X \geq 3$. Then there exists an ample vector bundle $\mathcal{E}$ on C of rank n such that $X = \mathbb{P}_S(\mathcal{E})$ and $L = H(\mathcal{E})$.
Then by [6, Example 3.9] we have

$$\mathrm{cl}_i(X, L) = \begin{cases} s_1(\mathcal{E}), & \text{if } i = 0, \\ 2g(C) - 2 + 2c_1(\mathcal{E}), & \text{if } i = 1, \\ c_1(\mathcal{E}), & \text{if } i = 2, \\ 0, & \text{if } i \geq 3. \end{cases}$$

In this case, $\mathrm{def}(X, L) = n - 2$ and $\mathrm{codeg}(X, L) = c_1(\mathcal{E})$.

(x) The case where (X, L) is $(\mathbb{P}_S(\mathcal{E}), H(\mathcal{E}))$, where S is a smooth surface and $\mathcal{E}$ is an ample vector bundle of rank $n - 1$. Then by [6, Example 3.10] we have

$$\mathrm{cl}_i(X, L) = \begin{cases} s_2(\mathcal{E}), & \text{if } i = 0, \\ (s_1(\mathcal{E}) + K_S)s_1(\mathcal{E}) + 2s_2(\mathcal{E}), & \text{if } i = 1, \\ c_2(S) + 3c_1(\mathcal{E})^2 + 2K_S c_1(\mathcal{E}), & \text{if } i = 2, \\ 2c_2(\mathcal{E}) + (c_1(\mathcal{E}) + K_S)c_1(\mathcal{E}), & \text{if } i = 3, \\ c_2(\mathcal{E}), & \text{if } i = 4 \text{ and } n \geq 4, \\ 0, & \text{if } i \geq 5 \text{ and } n \geq 5. \end{cases}$$

(x.1) Assume that $K_S + c_1(\mathcal{E})$ is not nef. Here we note that $\mathrm{rank}\, \mathcal{E} \geq 2 = \dim S$. Then by a result of [9, Theorem 1] we see that $(S, \mathcal{E}) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(1))$. In this case, $c_2(S) = 3$, $c_1(\mathcal{E})^2 = 4$, $K_S c_1(\mathcal{E}) = -6$, $c_2(\mathcal{E}) = 1$, $s_2(\mathcal{E}) = 3$. So we get the following.

$$\mathrm{cl}_i(X, L) = \begin{cases} 3, & \text{if } i = 0, \\ 4, & \text{if } i = 1, \\ 3, & \text{if } i = 2, \\ 0, & \text{if } i = 3. \end{cases}$$

Hence in this case $\mathrm{def}(X, L) = 1$ and $\mathrm{codeg}(X, L) = 3$.

Remark 2.1 Here we note that if $(S, \mathcal{E}) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(1))$, then $(X, L) = (\mathbb{P}_S(\mathcal{E}), H(\mathcal{E}))$ is a scroll over $\mathbb{P}^1$.

(x.2) Next we consider the case where $K_S + c_1(\mathcal{E})$ is nef. Then the following holds.

Claim 2.1 *$cl_i(X, L) > 0$ for every $0 \leq i \leq \min\{4, n\}$.*

Proof. First of all, since $\mathcal{E}$ is ample, we see from [7, Example 12.1.7] and Remark 1.1 that $\mathrm{cl}_0(X, L) = s_2(\mathcal{E}) > 0$. Next we consider the case of $i \geq 1$. $(K_S + c_1(\mathcal{E}))c_1(\mathcal{E}) \geq 0$ because $K_S + c_1(\mathcal{E})$ is nef. Moreover $c_2(\mathcal{E}) > 0$ since $\mathcal{E}$ is ample. Hence $\mathrm{cl}_1(X, L) > 0$, $\mathrm{cl}_3(X, L) > 0$ and $\mathrm{cl}_4(X, L) > 0$ for $n \geq 4$. (Here we note that $c_1(\mathcal{E}) = s_1(\mathcal{E})$.) Finally we consider the case of $\mathrm{cl}_2(X, L)$. We note the following.

(a) If $\kappa(S) \geq 0$, then $c_2(S) \geq 0$.

(b) If $\kappa(S) = -\infty$ and $q(S) = 0$, then $c_2(S) \geq 3$.

(c) If $\kappa(S) = -\infty$ and $q(S) \geq 1$, then $c_2(S) \geq 4(1 - q(S))$.

So if $\kappa(S) \geq 0$ or $\kappa(S) = -\infty$ and $q(S) = 0$, then

$$\begin{aligned} \mathrm{cl}_2(X, L) &= c_2(S) + 3c_1(\mathcal{E})^2 + 2K_S c_1(\mathcal{E}) \\ &\geq c_1(\mathcal{E})^2 > 0. \end{aligned}$$

If $\kappa(S) = -\infty$ and $q(S) \geq 1$, then

$$\begin{aligned} \mathrm{cl}_2(X, L) &= c_2(S) + 3c_1(\mathcal{E})^2 + 2K_S c_1(\mathcal{E}) \\ &\geq c_1(\mathcal{E})^2 + 4(g(S, c_1(\mathcal{E})) - q(S)). \end{aligned}$$

Since $\kappa(S) = -\infty$, we have $g(S, c_1(\mathcal{E})) \geq q(S)$ by [3, Theorem 2.1]. Therefore we get $\mathrm{cl}_2(X, L) \geq c_1(\mathcal{E})^2 > 0$. □

Therefore, in this case, we get $\mathrm{def}(X, L) = \max\{0, 4 - n\}$ and

$$\mathrm{codeg}(X, L) = \begin{cases} 2c_2(\mathcal{E}) + (c_1(\mathcal{E}) + K_S)c_1(\mathcal{E}), & \text{if } n = 3, \\ c_2(\mathcal{E}), & \text{if } n \geq 4. \end{cases}$$

In general, if X is a projective bundle over a smooth projective variety Y of dimension m with $\dim X \geq 2m$ and L is the tautological line bundle $H(\mathcal{E})$, then we can calculate $\mathrm{def}(X, L)$ and $\mathrm{codeg}(X, L)$.

Proposition 2.4 *Let X be an n-dimensional projective bundle $P_Y(\mathcal{E})$ over a smooth projective variety Y of dimension m and let $H(\mathcal{E})$ be the tautological line bundle. Assume that $n \geq 2m$. Then* $\mathrm{def}(X, H(\mathcal{E})) = n - 2m$ *and* $\mathrm{codeg}(X, H(\mathcal{E})) = c_m(\mathcal{E})$.

Proof. If $j - 2 \geq 2m - 1$, that is, $j \geq 2m + 1$, then by [5, Theorem 3.1 (3.1.1)] we have

$$\begin{aligned} \mathrm{cl}_j(P_Y(\mathcal{E}), H(\mathcal{E})) &= (-1)^j(e_j(P_Y(\mathcal{E}), H(\mathcal{E})) - 2e_{j-1}(P_Y(\mathcal{E}), H(\mathcal{E})) + e_{j-2}(P_Y(\mathcal{E}), H(\mathcal{E}))) \\ &= (-1)^j((j - m + 1)c_m(Y) - 2(j - m)c_m(Y) + (j - m - 1)c_m(Y)) \\ &= 0. \end{aligned}$$

If $j = 2m$, then by [5, Theorem 3.1 (3.1.1) and (3.1.2)]

$$\begin{aligned} \mathrm{cl}_{2m}(P_Y(\mathcal{E}), H(\mathcal{E})) &= (-1)^{2m}(e_{2m}(P_Y(\mathcal{E}), H(\mathcal{E})) - 2e_{2m-1}(P_Y(\mathcal{E}), H(\mathcal{E})) + e_{2m-2}(P_Y(\mathcal{E}), H(\mathcal{E}))) \\ &= ((m + 1)c_m(Y) - 2mc_m(Y) + (m - 1)c_m(Y) + c_m(\mathcal{E})) \\ &= c_m(\mathcal{E}) > 0. \end{aligned}$$

Hence by Definition 1.2 we have

$$\begin{aligned} \mathrm{def}(X, H(\mathcal{E})) &= \min\{\, i \mid \mathrm{cl}_{n-i}(X, H(\mathcal{E})) \neq 0\} = n - 2m. \\ \mathrm{codeg}(X, H(\mathcal{E})) &= c_m(\mathcal{E}). \end{aligned}$$

This completes the proof. □

Assume that (X, L) is a $\mathbb{P}^{n-3}$-bundle over a smooth projective variety Y with $n \geq 4$ and $\dim Y = 3$. Let $\mathcal{E}$ be an ample vector bundle on Y such that $X \cong \mathbb{P}_Y(\mathcal{E})$ and $L = H(\mathcal{E})$. Then by [5, Theorem 3.1] $\mathrm{cl}_i(X, L)$ is the following.

$$\mathrm{cl}_i(X,L) = \begin{cases} s_3(\mathcal{E}), & \text{if } i=0, \\ 3s_3(\mathcal{E}) + (s_1(\mathcal{E})+K_Y)s_2(\mathcal{E}), & \text{if } i=1, \\ 3s_3(\mathcal{E}) + 12(s_1(\mathcal{E})+K_Y)s_2(\mathcal{E}) \\ +(s_1(\mathcal{E})+K_Y)s_1(\mathcal{E})^2 + c_2(Y)s_1(\mathcal{E}), & \text{if } i=2, \\ -c_3(Y) + 2c_3(\mathcal{E}) - 2c_1(\mathcal{E})c_2(\mathcal{E}) + 4c_1(\mathcal{E})^3 \\ +3K_Y c_1(\mathcal{E})^2 + 2c_2(Y)c_1(\mathcal{E}), & \text{if } i=3, \\ 3c_3(\mathcal{E}) + 12(c_1(\mathcal{E})+K_Y)c_2(\mathcal{E}) \\ +(c_1(\mathcal{E})+K_Y)c_1(\mathcal{E})^2 + c_2(Y)c_1(\mathcal{E}), & \text{if } i=4, \\ 3c_3(\mathcal{E}) + (c_1(\mathcal{E})+K_Y)c_2(\mathcal{E}), & \text{if } i=5 \text{ and } n\geq 5, \\ c_3(\mathcal{E}), & \text{if } i=6 \text{ and } n\geq 6, \\ 0, & \text{if } i\geq 7 \text{ and } n\geq 7. \end{cases}$$

By considering the above results, we can propose the following conjecture.

Conjecture 2.1 *Assume that (X,L) is a $\mathbb{P}^{n-m}$-bundle over a smooth projective variety Y with $\dim Y = m$. Let $\mathcal{E}$ be an ample vector bundle on Y such that $X \cong \mathbb{P}_Y(\mathcal{E})$ and $L = H(\mathcal{E})$. Assume that $n \geq 2m$. For any integer i with $0 \leq i \leq m$ we set*

$$F_i(s_1(\mathcal{E}),\ldots,s_m(\mathcal{E})) := \mathrm{cl}_i(X,L).$$

Then for any integer j with $m \leq j \leq 2m$ we have

$$\mathrm{cl}_j(X,L) = F_{2m-j}(c_1(\mathcal{E}),\ldots,c_m(\mathcal{E})).$$

In particular

$$F_m(s_1(\mathcal{E}),\ldots,s_m(\mathcal{E})) = F_m(c_1(\mathcal{E}),\ldots,c_m(\mathcal{E})).$$

Remark 2.2 This conjecture is true for the case where $m = 1$, 2 and 3.

By looking at the above examples, we see that $\mathrm{cl}_{i+1}(X,L) = 0$ if $\mathrm{cl}_i(X,L) = 0$. So we can propose the following problem.

Problem 2.1 *Let (X,L) be a polarized manifold of dimension n and let i be an integer with $0 \leq i \leq n-1$. Is it true that $\mathrm{cl}_{i+1}(X,L) = 0$ if $\mathrm{cl}_i(X,L) = 0$?*

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