A GROTHENDIECK TOPOLOGY ON A SUBCATEGORY OF OPPOSITE CATEGORY OF NON COMMUTATIVE ALGEBRAS

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1. Introduction

The purpose of this paper is to show an existence of a Grothendieck topology on a subcategory of the opposite category of non commutative algebras. To define a Grothendieck topology is to define a way to "glue" objects, which will make it possible to define a non-commutative geometric object which might be called non commutative scheme.

The term "non commutative scheme" already appears in [5] in a slight different context, in which a non commutative scheme is defined as a ringed space. Our categorical approach provides another look at non commutative algebraic geometry. It in particular requires (and therefore offers some motivation for) a study on free products. In either way, non commutative algebraic geometry seems to play the same role in non commutative geometry (developed for example by

Conne ([2][3])) as algebraic geometry plays in commutative geometry. It will provide for example singularity theories, homological methods, and also some concrete examples of non commutative manifolds.

Our point of view in this paper is as follows.

(1) We basically want to regard the opposite of the category (algebra) of (unital associative) algebras as a geometric object.
(2) We define a class of homomorphisms which we call "bi-flat epimorphisms" They play a role which localizations play in the commutative case.
(3) We notice that a "base extension" of a good homomorphism may be bad (Counter example 6.1). Therefore we propose we pass to a subcategory $\mathcal{C}$ and consider only those "continuous maps" there. (Note that this does not mean we give up considering "discontinuous maps". See below.) Then we show that $\mathcal{C}$ has a Grothendieck topology (Main theorem 9.7).
(4) We show that certain class of homomorphisms (which we call bi-faithful "flaky" homomorphisms in this paper) have descent property. This implies that the set of "affine object valued maps", continuous or not, form a sheaf on the site $\mathcal{C}$ (Main theorem 9.8) .

We may make an analogy between this situation and a definition of measurable maps on a topological manifold. Although "gluing maps" should be continuous, we may consider any measurable maps on a manifold.

To make the idea more rigorous, let us begin with a review of the definition of Grothendieck topology. It is a generalization of the notion of "coverings" in the category of topological spaces.

Definition 1.1. (We essentially repeat the words used in [7]) We say a Grothendieck topology on a category $\mathcal{C}$ is given if we are given a set Cov of families $\{U_i \to U; i \in I\}$ of maps in $\mathcal{C}$ (called coverings) which satisfies the following axioms

(GT 1) If ϕ is an isomorphism then $\{\phi\} \in$ Cov.
(GT 2) If $\{U_i \to U\} \in$ Cov and $\{V_{ij} \to U_i\} \in$ Cov for each i then the family $\{V_{ij}\}$ obtained by composition is in Cov.
(GT 3) If $\{U_i \to U\} \in$ Cov and $V \to U$ is arbitrary morphism of $\mathcal{C}$, then the fiber product $U_i \times_U V$ exists and $\{U_i \times_U V \to V\} \in$ Cov.

A category equipped with a Grothendieck topology is called a site.

For example, the category (Top) of topological spaces with "open coverings" in the usual sense forms a site. A basic idea of non commutative geometry is to regard each non commutative algebra A as a "function ring" over a geometric object $\mathrm{Geo}(A)$. A category of all such objects would be the opposite $(\text{algebra})^{\text{opp}}$ of the category of algebras. For the reason (2) above we pass to a subcategory $\mathcal{C}$ of $(\text{algebra})^{\text{opp}}$. Our first main result then states

Theorem 9.7 A subcategory $\mathcal{C}$ of $(\text{non commutative algebras})^{\text{opp}}$ has a Grothendieck topology.

This result makes us possible to "glue" certain functors. It is explained in terms of sheaves on a site.

Definition 1.2. (We follow [7] again.) A sheaf of sets on a site is a contravariant functor $\mathcal{C} \to (\text{Sets})$ such that for any covering $U \to X$ of an object X in the category $\mathcal{C}$ we have an exact sequence of sets

$$F(X) \to F(U) \rightrightarrows F(U \times_X U).$$

This mechanism may be regarded as a generalization of gluing lemma for continuous maps on a topological space: Continuous maps defined on a open covering give rise to a continuous map defined on the original space if they coincide on intersections. We associate each object $\mathrm{Geo}(A)$ of $\mathcal{C}$ with "$\mathrm{Geo}(A)$-valued maps" functor

$$F(\mathrm{Geo}(B)) = \mathrm{Hom}_{\text{algebra}}(A; B).$$

Our second main result states

Theorem 9.8 For any algebra A, "$\mathrm{Geo}(A)$-valued maps" form a sheaf on the site $\mathcal{C}$.

Thus we may glue these "affine objects" together as a sheaf to make a "non commutative scheme".

Technically speaking, there are basically two points we emphasize in this paper. One is to employ "bi-flat epimorphism" (Definition 6.1) as a non-commutative version of "localization". (It is worthwhile to mention that "left-flat epimorphisms" play an important role in [5] and is refered to as "perfect homomorphism" there.) The other is to employ free product (or its mollified version (Definition 9.2)), rather than tensor product (employed for example in [9]), as an object corresponding to a product in the geometric sense.

As the result of this paper may be regarded as a step to define non commutative algebraic geometry, the author attempted to keep this paper self contained.

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He also thanks Masaki Maruyama, Akira Ishii and Kōta Yoshioka for discussions on algebraic geometry.

2. Non commutative polynomial algebras

All the rings and homomorphisms which appear in this paper are assumed to be unital. We fix a ground ring R (not necessarily commutative). It does not seem there exist a standard notation for non commutative polynomial algebras. We employ in this paper the following notation

Definition 2.1. Let A be a R-algebra. Then the algebra generated freely by A and indeterminates $\{X_\lambda\}_{\lambda\in\Lambda}$, modulo conditions

$$cX_\lambda = X_\lambda c \quad (\text{for all } c \in R,\ \lambda \in \Lambda)$$

is called "the polynomial algebra over A with coefficient ring R and indeterminates $\{X_\lambda\}$" and is denoted as

$$A[^R\{X_\lambda; \lambda \in \Lambda\}].$$

As in the commutative case, there is a natural grading on the polynomial algebra.

$$A[^R\{X_\lambda; \lambda \in \Lambda\}] = \bigoplus_d (A[^R\{X_\lambda; \lambda \in \Lambda\}])_d = \bigoplus_d \{\text{homogeneous polynomials of degree } d\}.$$

We have the following expression for a polynomial algebra with one indeterminate.

Lemma 2.1. *There is an isomorphism*

$$A[^R X] \cong \bigoplus_{i=0}^{\infty} (A \otimes_R A \otimes_R A \otimes_R \ldots \otimes_R A) = \bigoplus_{i=0}^{\infty} (A^{\otimes_R (i+1)})$$

as graded algebras, where the multiplication of the right hand side is defined by

$$(a_0 \otimes a_1 \otimes a_2 \otimes \ldots \otimes a_i) \times (b_0 \otimes b_1 \otimes b_2 \otimes \ldots \otimes b_j) = a_0 \otimes a_1 \otimes a_2 \otimes \ldots \otimes (a_i \times b_0) \otimes b_1 \otimes b_2 \otimes \ldots \otimes b_j$$

Proof. In fact, the isomorphism is obtained by replacing the indeterminate X by tensor sign "$\otimes$". Namely,

$$a_0 X a_1 X \ldots X a_n \mapsto a_0 \otimes a_1 \otimes \ldots \otimes a_n.$$

□

We use the following lemma, which guarantees us an existence of certain odd derivations.

Lemma 2.2. *Let F be an odd element of $P = A[^R X]$. That means,*

$$F \in \bigoplus_{n:\mathrm{odd}} P_n.$$

Then there exists a unique odd derivation δ_F on P such that

$$\delta_F(X) = F, \quad \delta_F(a) = 0 \quad \text{for all } a \in A.$$

(Recall that a odd derivation on P is an additive map δ such that the condition

$$\delta(FG) = \delta(F)G + (-1)^k F\delta(G)$$

holds for any $F \in P_n, G \in P$.)

3. Hochschild cohomology

We recall here briefly a definition of the Hochschild cohomology. Our definition coincides with the one given in [8], and is slightly different to that given in [1]. The two definitions coincide if the ground ring R is a commutative field, which is one of the most important cases. Let $I^\bullet(A)$ be a complex defined as

$$I^n(A) = (A[^R X])_{n+1} \quad (n \geq 0)$$

with the odd differential δ obtained as the restriction of the differential δ_X (δ_F with $F = X$) given in Lemma 2.2. Then we define

Definition 3.1. The Hochschild cohomology of a R-algebra A with coefficient in a A-bimodule M is the $Z(R)$(= the center of R)-module

$$H^n(A; M) = H^n(\mathrm{Hom}^\bullet_{A\text{-}\mathrm{bimod}}(I^\bullet(A), M)).$$

Note that our complex $I^\bullet$ here coincides with the complex $\beta(A, A)$ given in [8]. We have the following expressions for the zeroth and first Hochschild cohomology groups.

$$H^0(A; M) \cong M^A = \{m \in M; am = ma \text{ for all } a \in A\}$$

$$H^1(A; M) \cong \mathrm{Der}(A; M)/\mathrm{Inn}(A; M) = \frac{\{A \to M; \text{R-derivation}\}}{\{A \to M; \text{Inner R-derivation}\}}$$

We recall another notion which we will use later. The "forgetful" functor

$$\square : (A\text{-}\mathrm{bimod}) \to (R\text{-}\mathrm{bimod})$$

has a left adjoint F defined by

$$F(M) = A \otimes_R M \otimes_R A,$$

So we have a resolvent pair in the sense of [8], denoted $\mathcal{R}(A\text{- bimod}, R\text{- bimod})$. The Hochschild cohomology groups may be expressed as relative extension groups as follows.

$$H^i(A; M) = \operatorname{Ext}^i_{\mathcal{R}(A\text{- bimod}, R\text{- bimod})}(A; M)$$

See [8] for details.

4. Free products

The free product, or amalgamation, plays the role of the co-fiber-product in the category of non commutative algebras. We remind the reader that in this paper we always assume our rings and morphisms are all unital. Accordingly, "free products" in this paper means unital ones. We recall the definition for the purpose of later references.

Definition 4.1. Let B, C be A-algebras. Then the free product $B *_A C$ of B and C over A is the A-algebra together with two "canonical" A-homomorphisms

$$\iota_1 : B \to B *_A C$$
$$\iota_2 : C \to B *_A C$$

such that, for any other A-algebra D with two A-homomorphisms

$$\varphi_1 : B \to D$$
$$\varphi_2 : C \to D,$$

there exists a unique A-homomorphism, denoted (φ_1, φ_2) from $B *_A C$ to D such that we have

$$(\varphi_1, \varphi_2) \circ \iota_i = \varphi_i \quad (i = 1, 2).$$

A good deal of problems occur when dealing with free products since the free products, unlike tensor products, do not behave bilinearly with respect to B, C. To improve this, we will introduce a mollified free product for special class of algebras (Definition 9.2). We end this section by stating the following easy lemma relating tensor product and free product in a case when $B = C$.

Lemma 4.1. *A module homomorphism*

$$B \otimes_A B \to B *_A B$$

mapping $b_1 \otimes b_2$ *to* $\iota_1(b_1)\iota_2(b_2)$ *is injective, and the image is a direct summand of* $B *_A B$.

PROOF. Put

$$I_\Delta = \ker(B *_A B \stackrel{(\mathrm{id},\mathrm{id})}{\to} B),$$

$$\mathfrak{M}_\Delta = \ker(B \otimes_A B \stackrel{\text{multiplication}}{\to} B).$$

Then we have

(1) $(B *_A B)/I_\Delta \cong B \otimes_A B/\mathfrak{M}_\Delta \cong B$.
(2) Both A-bimodules I_Δ/I_Δ^2 and $\mathfrak{M}_\Delta$ serves as the non-commutative Kähler differentials of B over A. (Hence they are isomorphic.)

Thus we conclude

$$(B *_A B)/I_\Delta^2 \cong B \otimes_A B.$$

$\square$

5. FAITHFULLY FLATNESS

In this section we review basic facts about faithfully flatness. All results may be proved in analogous ways to the commutative case. We first recall the definition of flatness.

Definition 5.1. Let A be an R-algebra. An A-algebra C is called left-flat (respectively, right-flat) over A if the functor $C \otimes_A$ (respectively, $\otimes_A C$) is an exact functor. It is called bi-flat if it is left- and right-flat.

Lemma 5.1. *Let C be a left-flat A-algebra. Then the followings are equivalent.*

(1) *For any non-zero left A-module M, we have*

$$C \otimes_A M \neq 0.$$

(2) *For any sequence*

$$0 \to L \to M \to N \to 0$$

of left A-module, if a sequence

$$0 \to C \otimes_A L \to C \otimes_A M \to C \otimes_A N \to 0$$

obtained by tensoring C to the original sequence from the left is exact, then the original sequence is also exact.

(3) *For any A-module M, A module homomorphism*

$$M \ni m \mapsto 1 \otimes m \in C \otimes_A M$$

is injective.

(4) *For any left ideal I of A we have*

$$B \otimes_A (A/I) \neq 0$$

Definition 5.2. If one (hence all) of the conditions in Lemma 5.1 holds, we call C an algebra left-faithfully flat over A, and the structure homomorphism $A \to C$ is called left-faithfully flat homomorphism. Right-faithfully flatness is defined in a similar manner. An algebra or a homomorphism is called bi-faithfully flat if it is left- and right-faithfully flat.

Lemma 5.2. *Left faithfully flat morphisms are injective.*

6. Bi-flat epimorphisms

In this section we introduce bi-flat epimorphisms, which will be a central notion in this paper. This notion is essentially known to specialists. In fact, in [5] one sees that left-flat epimorphisms play an important role in the theory developed there. Here we examine their behaviour with respect to free product.

We use the term "epimorphism" in the category theoretical sense. We first note the following lemma.

Lemma 6.1. *For a homomorphism $\varphi : A \to B$, the followings are equivalent*

(1) *We have an isomorphism $B \otimes_A B \cong B$ of B-bimodules via the product map $b_1 \otimes b_2 \mapsto b_1 b_2$.*
(2) *We have an isomorphism $B *_A B \cong B$ of algebras via the codiagonal map (id, id)*
(3) *φ is a (category theoretically) epimorphism. That means, for any A-algebra C, an A-homomorphism from B to C, if there exists one, is unique.*

Proof. That (2) implies (1) follows from Lemma 4.1. To prove the converse implication, we note that canonical maps $\iota_1, \iota_2 : B \to B *_A B$ factors through $B \otimes_A B \to B *_A B$, so that we have in fact $\iota_1 = \iota_2$. The equivalence of (2) and (3) follows from the universality of free products. □

Definition 6.1. If one (hence all) of the conditions in Lemma 6.1 is satisfied, the homomorphism φ is called an epimorphism. It is called a bi-flat epimorphism (BE) if it is an epimorphism and is bi-flat. If the homomorphism φ referring to is clear, we simply say, by abuse of language, that B is BE over A.

Note that epimorphisms need not be surjective maps in the set theoretic sense. We distinguish in this paper the term "epimorphism" from the term "surjective map". The former is used in the categorical sense as above while the latter is used in the set-theoretical sense.

Bi-flat epimorphisms play similar role in a theory of non commutative algebras as localization does in the theory of commutative algebras.

Examples .

(1) Identity maps are bi-flat epimorphisms.
(2) Every localization of a commutative algebras in the usual sense is a bi-flat epimorphism.

Definition 6.2. Let A be an algebra and S be a subset of A. Then the localization A_S of A with respect to S is defined as follows.

$$A_S = A[^R\{X_s; s \in S\}]/X_s s = sX_s = 1$$

We also refer to the natural map $A \to A_S$ as localization.

Remark 6.1. A localization A_S by a set $S = \{s\}$ consisting of a single element is a "base extension" of a flat homomorphism

$$R[^R X] \to R[^R X, 1/X]$$

by a homomorphism

$$R[^R X] \to A : \quad X \mapsto s.$$

That means,

$$A_S \cong A *_{R[^R X]} R[^R X, 1/X].$$

Lemma 6.2. *Any localization $A \to A_S$ of an algebra is an epimorphism.*

Proof. Any element in $A_S \otimes_A A_S$ is a linear combination of elements which look like

$$t \otimes a_0 s_1^{-1} a_1 \dots s_n^{-1} a_n,$$

where $t \in A_S, a_i \in A, s_i \in S$. But we have a computation

$$\begin{aligned} &t \otimes a_0 s_1^{-1} a_1 \dots s_n^{-1} a_n \\ =&ta_0 \otimes s_1^{-1} a_1 \dots s_n^{-1} a_n \\ =&ta_0 s_1^{-1} s_1 \otimes s_1^{-1} a_1 \dots s_n^{-1} a_n \\ =&ta_0 s_1^{-1} \otimes a_1 \dots s_n^{-1} a_n \\ =&\dots \\ =&ta_0 s_1^{-1} a_1 \dots s_n^{-1} a_n \otimes 1, \end{aligned}$$

which shows that a splitting $\otimes_1 : A_S \to A_S \otimes_A A_S$ of the multiplication map $A_S \otimes_A A_S \to A_S$ is surjective. □

We note that localizations of non commutative algebras need not be bi-flat, as the following counter example indicates.

Counter Example 6.1. Let

$$A = R[^R x, t]/tx = (x+1)t, \quad S = \{x\}.$$

Then $A \to A_S = A_x$ is not bi-flat. In fact, if we let

$$M = \bigoplus_{n \in \mathbb{N}} \mathbb{C} e_n$$

$$N = \bigoplus_{n \in \mathbb{N} \setminus \{0\}} \mathbb{C} e_n$$

be direct sum of one dimensional vector spaces and define action of A on M obtained by

$$x(e_n) = -n e_n, \quad t(e_n) = e_{n+1},$$

then N is stable under this action. But we easily see the followings.

(1) $A_x \otimes_A M = 0$,
(2) $A_x \otimes_A N = N \neq 0$.

This shows that A_x is not left-flat. In particular, in view of Remark 6.1 we see that a "base extension" of a flat homomorphism may not be flat.

However, we have the following result of M. Kashiwara. ([6])

Lemma 6.3. *If a subset S of an algebra A satisfies the following properties, then the localization A_S is left-flat over A.*

(1) *The set S is closed under multiplication and contains 1.*
(2) *For any elements $a \in A$ and $s \in S$, there exist elements $b \in A$, $t \in S$ such that we have*

$$ta = bs.$$

(3) *If furthermore in the above situation a is an element of S, then we may choose b, t as above with $b \in S$*
(4) *If $a \in A, s \in S$ satisfies $as = 0$, then there exists an element $t \in S$ such that $ta = 0$.*

We have an analogous sufficient condition for right flatness.

Proof. For any A-module M, we have under the conditions of the lemma, an description of localization

$$A_S = S^{-1}M = S \times M / \sim,$$

where the equivalence relation $\sim$ is defined as follows

$$(a, m) \sim (b, n) \Leftrightarrow \exists s, t \in S; sb = ta, sn = tm.$$

□

The author does not know if compositions of localizations are always localization again. But we have the following lemma.

Lemma 6.4. *A composition of two bi-flat epimorphisms $A \to B$, $B \to C$ is also a bi-flat epimorphism.*

The proof is obvious. $\square$

Thus the class of all algebras with bi-flat epimorphisms forms a subcategory of the category of algebras. We will later see that free products serves as coproducts in this category too.

Lemma 6.5. *Let $A \to B$ be an epimorphism. Then for any B-left module M_1 and right B-module M_2, we have*

$$M_2 \otimes_A M_1 \cong M_2 \otimes_B M_1$$

Proof.

$$M_2 \otimes_A M_1 \cong (M_2 \otimes_B B) \otimes_A (B \otimes_B M_1) \cong M_2 \otimes_B (B \otimes_A B) \otimes_B M_1 \cong M_2 \otimes_B M_1$$

$\square$

Corollary 6.6. *Let $\psi : B \to C$ be a homomorphism, and $\varphi : A \to B$ be an epimorphism. Assume that C is left (respectively, right)-flat over A. Then C is also left (respectively, right)-flat over B. In particular, if both φ and $\psi \circ \varphi$ are bi-flat epimorphisms, then so is ψ.*

Proof. The first claim follows easily from the Lemma. Flatness of ψ follows from the first claim. That ψ is an epimorphism is verified as follows.

$$C \otimes_B C \cong C \otimes_A C \cong C.$$

$\square$

Lemma 6.7. *Let A be an algebra, B, C two A-algebras which are BE over A. Then the free product $B *_A C$ is also BE over A. Thus it is also BE over B and over C.*

Proof. Using Lemma 6.1, we obtain an isomorphism

$$(B *_A C) *_A (B *_A C) \cong (B *_A B) *_A (C *_A C) \cong B *_A C.$$

Thus we see that $B *_A C$ is also BE over A. To prove the flatness, we use an explicit description for the free product using tensor products. Namely, we have the following lemma, which tells us that the free product in our case is actually an inductive limit of bi-flat modules, and hence is also flat. The last line of the statement is deduced from the rest of the lemma by using Corollary 6.6. $\square$

Lemma 6.8. *Under the assumption of Lemma 6.7, we have an inductive limit expression of the free product*

$$B *_A C \cong \varinjlim (B \otimes_A C)^{\otimes_A n}.$$

The connecting map for the inductive limit in the right hand side is given by

$$M^{\otimes_A n} \ni m \mapsto m \otimes u \in M^{\otimes_A n} \otimes_A M,$$

where we put $M = (B \otimes_A C)$, and the element u of M is given by

$$u = 1_B \otimes 1_C.$$

PROOF. We note that for any two elements b_1, b_2 in B, we have

$$b_1 \otimes 1_C \otimes b_2 = b_1 b_2 \otimes 1_C \otimes 1_B,$$

where both hand sides are considered as elements of $B \otimes_A C \otimes_A B$. To see this, we observe that we have a well-defined map

$$B \otimes_A B \to B \otimes_A C \otimes_A B$$

defined by

$$b_1 \otimes b_2 \mapsto b_1 \otimes 1_C \otimes b_2,$$

and recall that $B \otimes_A B \cong B$. Using this identity (and an analogous one with B and C interchanged), we observe firstly

$$x \otimes u = u \otimes x \text{ in } M \otimes_A M$$

for any x in M. Then secondly we see that the canonical identification map

$$M^{\otimes_A m} \otimes_A M^{\otimes_A n} \to M^{\otimes_A (m+n)}$$

gives a bilinear map compatible with the inductive system, which gives rise to a bilinear product

$$(\varinjlim_n M^{\otimes_A n}) \otimes_A (\varinjlim_n M^{\otimes_A n}) \to \varinjlim_n M^{\otimes_A n}$$

on the inductive limit. Thirdly we may verify that this product is associative, with the unit element u. We finally note that this algebra admits an obvious canonical homomorphism from both B and C, and that it has the universal property of the free product. □

Proposition 6.9. *The free product $B *_A C$ serves as the coproduct of B, C in the category of non commutative R-algebras with bi-flat epimorphisms.*

Proof. We have already proved in Lemma 6.7 that all maps appearing in the following commutative diagram are bi-flat epimorphisms.

$$\begin{array}{ccc} A & \longrightarrow & B \\ \big\downarrow & & \big\downarrow \\ C & \longrightarrow & B *_A C \end{array}.$$

Now let D be an A-algebra which is BE over A with A-homomorphisms $\alpha : B \to D$ and $\beta : C \to D$. (The homomorphism α, β are automatically bi-flat epimorphisms, in view of Corollary 6.6.) We need to prove that the map (α, β) is a bi-flat epimorphism. But this also follows from Lemma 6.6. □

7. Geometry of the set of left ideals

In this section we observe how the set of left ideals of an algebra behaves with respect to bi-flat epimorphisms.

We employ the following notations

$$\mathrm{Spl}(A) = \{\text{left ideal of A}\}, \tag{7.1}$$

$$\mathrm{Spr}(A) = \{\text{right ideal of A}\}. \tag{7.2}$$

Furthermore, for any homomorphism $\varphi : A \to B$ between algebras, we define "the associate maps"

$$\mathrm{Spl}(\varphi) : \mathrm{Spl}(B) \to \mathrm{Spl}(A), \quad \mathrm{Spr}(\varphi) : \mathrm{Spr}(B) \to \mathrm{Spr}(A)$$

defined by

$$\mathrm{Spl}(\varphi)(J) = \varphi^{-1}(J), \quad \mathrm{Spr}(\varphi)(J) = \varphi^{-1}(J).$$

It is easy to see the following.

Lemma 7.1. *A left ideal I of A is an image of* $\mathrm{Spl}(\varphi)$ *if and only if an identity*

$$\varphi^{-1}(B \centerdot \varphi(I)) = I. \tag{7.3}$$

holds.

We may use these spaces of ideals to see whether a homomorphism is faithfully flat. Namely, the following lemma holds.

Lemma 7.2. *Let $\varphi : A \to B$ be a left-flat homomorphism. Then the followings are equivalent.*

(1) *φ is left-faithfully flat.*
(2) *The associate map* $\mathrm{Spl}(\varphi) : \mathrm{Spl}(B) \to \mathrm{Spl}(A)$ *is surjective.*
(3) *The associate map* $\mathrm{Spl}(\varphi) : \mathrm{Spl}(B) \to \mathrm{Spl}(A)$ *maps surjectively on the set of maximal ideals in A.*

Proof. This follows easily from Lemma 5.1. □

Lemma 7.3. *If a homomorphism $\varphi : A \to B$ is a bi-flat epimorphism, then the induced map* $\mathrm{Spl}(\varphi)$ *is injective. To be more precise, we have an equality*

$$J = B \centerdot \varphi(\varphi^{-1}(J))$$

for any element J of $\mathrm{Spl}(B)$.

Proof. We have an exact sequence

$$0 \to A/\varphi^{-1}(J) \to B/J.$$

Tensoring B over A from the left of this sequence, we obtain the following exact sequence.

$$0 \to B/B\varphi(\varphi^{-1}(J)) \to B/J.$$

(Note that $B \otimes_A M \cong M$ for any left B-module M) □

Our next task is to investigate behaviors of $\mathrm{Spl}(\varphi)$. The result will be summarized in Proposition 7.6.

Lemma 7.4. *Let $\varphi : A \to B, \psi : A \to C$ be bi-flat epimorphisms. Then we have*

$$\mathrm{Spl}(B) \cap \mathrm{Spl}(C) = \mathrm{Spl}(B *_A C),$$

where both hand sides are considered as subsets of $\mathrm{Spl}(A)$.

Proof. Since we already know in Lemma 6.9, that the all maps appearing in a commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ \big\downarrow & & \big\downarrow \\ C & \longrightarrow & B *_A C \end{array} .$$

are bi-flat epimorphisms, we deduce we have an inclusion

$$\mathrm{Spl}(B *_A C) \subset \mathrm{Spl}(B) \cap \mathrm{Spl}(C).$$

To prove the converse inclusion, we take $I \in \mathrm{Spl}(A)$ which belongs to $\mathrm{Spl}(B) \cap \mathrm{Spl}(C)$. That means, we have equalities

$$\varphi^{-1}(B\varphi(I)) = I = \psi^{-1}(C\psi(I)),$$

or, equivalently, the following sequences are exact.

$$0 \to A/I \to B/B\varphi(I) \tag{7.4}$$

$$0 \to A/I \to C/C\psi(I) \tag{7.5}$$

We derive from the exact sequence 7.4, using the fact that C is bi-flat over A, another exact sequence

$$0 \to C \otimes_A (A/I) \to C \otimes_A (B/B\varphi(I)).$$

But the algebra $C \otimes_A (A/I) \cong C/C\psi(I)$ contains A/I inside (7.5), so that we have

$$0 \to A/I \to C \otimes_A (B/B\varphi(I))(\cong C \otimes_A B \otimes_A (A/I)) \quad \text{:exact.}$$

Tensoring B to the above sequence and continuing the above argument, we have

$$0 \to A/I \to B \otimes_A C \otimes_A B \otimes_A (A/I) \quad \text{:exact.}$$

Continuing the argument again and again we finally conclude that a sequence

$$0 \to A/I \to \varinjlim (C \otimes_A B)^{\otimes_A n} \otimes_A A/I$$

is exact. Now the lemma follows from the lemma 6.8. □

Lemma 7.5. *Let $\varphi : A \to B$ be a bi-flat epimorphism. If the associate map* $\mathrm{Spl}(\varphi) : \mathrm{Spl}(B) \to \mathrm{Spl}(A)$ *is surjective (hence bijective according to Lemma 7.3), then φ is an isomorphism.*

Proof. Since $\{0_A\} \in \mathrm{Spl}(A)$ is in the image of $\mathrm{Spl}(\varphi)$, we easily see that the homomorphism φ is injective. On the other hand, we have

$$B \otimes_A (B/\varphi(A)) \cong B/B\varphi(A) = 0,$$

which implies $B/\varphi(A) = 0$, since φ is faithfully flat (Lemma 5.1 (4) , equation (7.3)). Thus the homomorphism φ is surjective. □

Using Lemma 7.2, we may rephrase the above lemma as “a bi-flat epimorphism is left-faithful if and only if it is an isomorphism”.

Proposition 7.6. *A map*

$$\begin{aligned} &\{\textit{algebra which is BE over } A\ \}/\cong \\ \to &\{\textit{subset of } \mathrm{Spl}(A)\} \end{aligned}$$

determined by

$$B \mapsto \mathrm{Spl}(B),$$

where the right hand side is considered as a subset of $\mathrm{Spl}(A)$ *on the ground of the previous lemma, is injective. Furthermore, this injection has the following properties.*

(1) *Free products correspond to intersections via this injection.*
(2) *“There exists an A-homomorphism $B \to C$”* $\Leftrightarrow \mathrm{Spl}(C) \subset \mathrm{Spl}(B)$.

(Note that the A-homomorphism $B \to C$ in Claim (2) above is unique (Lemma 6.1)

Proof. Claim (1) is the content of Lemma 7.4. The " $\Longrightarrow$ " part of Claim (2) is clear. To prove the converse of the implication, we apply claim (1) to the algebra $B *_A C$ and see that the canonical homomorphism $C \to B *_A C$ satisfies the assumption of the previous lemma and is therefore an isomorphism. We thus obtain an A-homomorphism $B \to C$ by using the canonical homomorphisms as follows.

$$B \to B *_A C \cong C.$$

This completes the proof □

8. $\mathcal{F}$-étale map

Bi-flat epimorphisms satisfy some homological properties. To clarify this, we generalize in this section the notion of formally étale morphisms to the non commutative case.

Definition 8.1. Let A, B be two algebras, $\mathcal{F}$ a family of B-modules. A homomorphism $\varphi : A \to B$ is called $\mathcal{F}$-étale, if the associated map for Hochschild cohomology

$$H^i(\varphi) : H^i(B; M) \to H^i(A; M)$$

gives an isomorphism for any member M of $\mathcal{F}$ and for any i.

The choice of appropriate $\mathcal{F}$ for defining "genuine" étale maps seems to require some considerations. In this paper, we will employ the following important classes of modules as $\mathcal{F}$.

$$\mathcal{F}_a = \{\text{all } B\text{-bimodules}\} \tag{8.1}$$

$$\mathcal{F}_c = \{M; am = ma \text{ for all } m \in M, \text{ and for all } a \in Z(B)\} \tag{8.2}$$

(Here we denote by $Z(\bullet)$ the set of central elements of the algebra.) It follows readily from the definition that φ is $\mathcal{F}_a$-étale if and only if it is $\mathcal{F}$-étale for any class $\mathcal{F}$ of B-modules. If the homomorphism φ is bi-flat, we have the following lemma.

Lemma 8.1. *A bi-flat homomorphism $\varphi : A \to B$ is $\mathcal{F}$-étale if and only if the homomorphism*

$$\operatorname{Ext}^i_{\mathcal{R}(B\text{-}\mathrm{bimod}, R\text{-}\mathrm{bimod})}(B, N) \to \operatorname{Ext}^i_{\mathcal{R}(B\text{-}\mathrm{bimod}, R\text{-}\mathrm{bimod})}(B \otimes_A B, N)$$

between relative extension groups induced by the multiplication map

$$B \otimes_A B \to B$$

is bijective for any element $N \in \mathcal{F}$

The proof is standard and is left to the reader.

Remark 8.1. If the algebras A, B are commutative and the homomorphism φ is formally étale in the usual sense, then φ is $\mathcal{F}_c$-étale. In fact, the class $\mathcal{F}_c$ in this case is the class of "usual B-modules", that is B-bimodules for which the left multiplication and the right multiplication coincide. Then we may use the well-known fact that "the diagonal" B is a direct summand of the algebra $B \otimes_A B$ as an algebra.

Lemma 8.2. *Bi-flat epimorphisms are $\mathcal{F}$-étale for any $\mathcal{F}$.*

Proof. This is a direct consequence of Lemma 8.1. □

Corollary 8.3. *Let φ be a $\mathcal{F}_c$-étale homomorphism. Then φ maps central elements to central elements. In particular, bi-flat epimorphisms maps central elements to central elements.*

Proof. We put $i = 0, M = B$ in the definition of an $\mathcal{F}$-étale map and obtain an equation

$$B^A = B^B (= Z(B) \text{ (the center of B)}).$$

On the other hand, φ obviously maps central elements of A to elements in B^A. This completes the proof □

A motivation for defining $\mathcal{F}$-étale homomorphisms comes from the (infinitesimal) deformation theory of non commutative algebras ([4]). Let ϵ be a dual number ($\epsilon^2 = 0$) which commutes with everything. Recall that an infinitesimal deformation of an algebra A over a ring R is by definition given by introducing a $R[\epsilon]$-algebra structure on a $R[\epsilon]$-bimodule $A[\epsilon]$ which reduces to the original algebra when we let $\epsilon = 0$. More explicitly speaking, the algebra structure is given by introducing a multiplication law

$$m_\epsilon(f + \epsilon g, h + \epsilon k) = fh + \epsilon(fk + gh + \varphi(f, g)), \tag{8.3}$$

where $\varphi : A \times A \to A$ is a biadditive homomorphism which satisfies the following conditions.

$$\varphi(cf, g) = c\varphi(f, g),\ \varphi(fc, g) = \varphi(f, cg),\ \varphi(f, gc) = \varphi(f, g)c \quad (f, g \in A, c \in R)$$
$$f\varphi(g, h) - \varphi(fg, h) + \varphi(f, gh) - \varphi(f, g)h = 0 \quad (f, g, h \in A) \quad \text{(cocycle condition)}$$

We will refer to the deformed algebra with the multiplication law 8.3 as A_φ. We say that two such infinitesimal deformations given by cocycles φ, ψ are equivalent if and only if there exists a $R[\epsilon]$-isomorphism

$$A_\varphi \to A_\psi$$

which reduces to the identity when we let $\epsilon = 0$. The reader may easily verify that the equivalence classes of infinitesimal deformations are the same thing as extensions of A by A, and are thus parametrized by the cohomology group

$$H^2(A;A).$$

We may also define deformation of a homomorphism as a homomorphism between deformed algebras which reduces to the homomorphism between the original algebras.

An étale homomorphism behaves neatly with infinitesimal deformations, as the following lemma shows.

Lemma 8.4. *Assume $\iota : A \to B$ be a $\{B\}$-étale homomorphism. Then for all infinitesimal deformation φ of A, there exists an infinitesimal deformation ψ of B and an infinitesimal deformation $\tilde{\iota}$ of ι such that the following diagram commutes.*

$$\begin{array}{ccc} A_\varphi & \xrightarrow{\tilde{\iota}} & B_\psi \\ {\scriptstyle \epsilon\to 0}\downarrow & & \downarrow{\scriptstyle \epsilon\to 0} \\ A & \xrightarrow{\iota} & B \end{array}$$

Furthermore, $\tilde{\iota}$ is unique up to a conjugacy by an infinitesimal deformation $\widetilde{\mathrm{id}_B}$ of identity on B.

Proof. Existence is essentially ensured by an existence of a map $H^2(A;A) \to H^2(B;B)$ defined as

$$H^2(A;A) \overset{H^2(A;\iota)}{\to} H^2(A;B) \overset{H^2(\iota;B)}{\cong} H^2(B;B),$$

where $H^2(A;\iota)$ and $H^2(\iota;B)$ are additive maps associated to ι in the obvious way. We leave the rest to the reader. $\square$

9. The category $\mathcal{C}$

Definition 9.1. A homomorphism $\varphi : A \to B$ is called flaky homomorphism if it is a direct sum of bi-flat epimorphisms. Namely, we have a direct sum decomposition of B

$$B = \oplus_{i=1}^n B_i$$

into finite components $\{B_i\}$ such that each component φ_i of φ (with respect to this decomposition) is a bi-flat epimorphism.

Lemma 9.1. *Flaky homomorphisms are $\mathcal{F}_c$-étale. (Hence they map central elements to central elements.*

Proof. An easy homological argument shows that a direct sum of $\mathcal{F}_c$-étale maps is also $\mathcal{F}_c$-étale. The result now follows from this fact and Lemma 8.2. $\square$

Lemma 9.2. *A composition of two flaky homomorphisms are also flaky.*

PROOF. Since a direct sum decomposition of an algebra corresponds to a central idempotent, the result easily follows from the previous lemma. $\square$

The category $\mathcal{C}$ is defined as a subcategory of (Non commutative algebras)$^{\mathrm{opp}}$ as follows.

$$\begin{aligned}\mathrm{Object}(\mathcal{C}) &= \{R\text{-algebras}\}\\ \mathrm{Morphism}(\mathcal{C}) &= \{\text{opposite of flaky homomorphisms}\}\end{aligned}$$

To indicate that an algebra A is regarded as an object of $\mathcal{C}$, we employ the notation $\mathrm{Geo}(A)$. A morphism $\mathrm{Geo}(B) \to \mathrm{Geo}(A)$ thus corresponds to a flaky homomorphism $A \to B$.

This category has fiber products. To see this, we first introduce "mollified free products".

Definition 9.2. Let A be R-algebra, B,C two A-algebras. The mollified free product of B,C over A is defined as follows.

$$B \overset{\circ}{*}_A C = (B *_A C)/\langle [Z(B), C], [B, Z(C)] \rangle.$$

We note first the following lemma.

Lemma 9.3. *If two homomorphism $\varphi : A \to B, \psi : A \to C$ are bi-flat epimorphisms, then the mollified free product $B \overset{\circ}{*}_A C$ coincides with the free product $B *_A C$.*

PROOF. This follows from the fact that $B *_A C$ is BE over B, C (Lemma 6.7), and that bi-flat epimorphisms map central elements to central elements (Corollary 8.3). $\square$

Then we have the following result.

Lemma 9.4. *The category $\mathcal{C}$ has fiber products. They are given by mollified free products. That is,*

$$\mathrm{Geo}(B) \times_{\mathrm{Geo}(A)} \mathrm{Geo}(C) = \mathrm{Geo}(B \overset{\circ}{*}_A C).$$

Furthermore, if we have a decomposition

$$B = \bigoplus_{i=1}^{n} B_i,$$
$$C = \bigoplus_{j=1}^{m} C_j$$

such that the homomorphisms φ, ψ decompose as a sum of bi-flat epimorphisms with respect to these, then we have

$$B \overset{\circ}{*}_A C \cong \bigoplus_{i,j} B_i *_A C_j.$$

PROOF. Let $A \to B, A \to C$ be two flaky homomorphisms. Given two flaky A-homomorphisms $\alpha : B \to D, \beta : C \to D$, we have a well-defined homomorphism

$$B *_A C \to D,$$

since we have a universality for free products. But since α, β are flaky, images $\alpha(Z(B)), \beta(Z(C))$ of the centers are contained in the center $Z(D)$ of D. We thus have a homomorphism

$$B \overset{\circ}{*}_A C \to D.$$

It remains to prove that homomorphisms

$$B \to B \overset{\circ}{*}_A C, \quad C \to B \overset{\circ}{*}_A C, \quad B \overset{\circ}{*}_A C \to D$$

are all flaky. We reduce this problem to Lemmas 6.7 and 9.3 by dividing each algebras with central idempotents. □

Corollary 9.5. *If a homomorphism $A \to B$ is flaky, then a module homomorphism*

$$B \otimes_A B \ni b_1 \otimes b_2 \mapsto \iota_1(b_1)\iota_2(b_2) \in B \overset{\circ}{*}_A B$$

is injective. (Here homomorphisms ι_1, ι_2 are natural ones from B to $B \overset{\circ}{*}_A B$.)

PROOF. This is a consequence of the above Lemma and the Lemma 4.1. □

Lemma 9.6. *Let $\varphi : A \to B$, $\psi : A \to C$ be two flaky homomorphisms. If φ is left-faithfully flat, then the "base extension"*

$$C \to B \overset{\circ}{*}_A C$$

is also left-faithfully flat.

PROOF. Using Lemma 9.4 we may assume that the homomorphism ψ is a bi-flat epimorphism. Let

$$B = \bigoplus_{i=1}^{n} B_i$$

be a direct sum decomposition with respect to which the homomorphism ϕ decomposes into a sum of bi-flat epimorphisms. Take a maximal ideal J of C. Since $\mathrm{Spl}(\phi)$ is surjective, one may find ideals $\{J_i\}_{i=1}^n$ of B_i such that

$$\phi^{-1}(\bigoplus_{i=1}^{n} J_i)(=\cap_i(\phi^{-1}(J_i)) = \psi^{-1}(J) \tag{9.1}$$

holds. Using the flatness of ψ, we deduce from the above equation the following.

$$C\psi(\cap_i(\phi^{-1}(J_i))) = C\psi(\psi^{-1}(J)) = J(\text{Lemma 7.3})$$

Using the maximality of J, we conclude that there exist i_0 such that

$$J = C\psi(\phi^{-1}(J_{i_0})) \tag{9.2}$$

holds. Equations 9.1, 9.2 implies in particular, an equality

$$\psi^{-1}(J) = \phi^{-1}(J_{i_0}),$$

which shows that J is an element of the set $\mathrm{Spl}(B_{i_0}) \cap \mathrm{Spl}(C) (\subset \mathrm{Spl}(A))$. □

We now state the main theorem of this paper.

Theorem 9.7. *The category $\mathcal{C}$ admits a Grothendieck topology. A covering map is given by a finite set of homomorphisms*

$$\{\mathrm{Geo}(\varphi_i) : \mathrm{Geo}(B_i) \to \mathrm{Geo}(A); i = 1, \dots, n\}$$

such that,

(1) *Each homomorphism φ_i is flaky,*
(2) *The homomorphism $A \to \oplus B_i$ obtained from φ_i 's is bi-faithful.*

Proof. (GT1) of Definition 1.1 is obvious. (GT2) results from Lemma 9.2 and a fact that a composition of faithful homomorphisms are faithful. (GT3) is a result of Lemma 9.6.

This is an easy consequence of Lemma 9.4. □

Theorem 9.8. *For any algebra S over the ground ring R, the functor*

$$A \mapsto \mathrm{Hom}_{R\text{-}algebra}(A, S)$$

is a sheaf over the site $\mathcal{C}$. In other words, the sequence

$$A \to B \underset{\iota_2}{\overset{\iota_1}{\rightrightarrows}} B \overset{\circ}{*}_A B$$

is exact. (That is, A is the difference kernel of ι_1, ι_2.)

Proof. The proof is essentially identical with the commutative case. (We already know in Lemma 5.2 that the homomorphism $A \to B$ is injective.) For the proof of the exactness of the above sequence, it is enough to prove the exactness of

$$A \to B \rightrightarrows B \otimes_A B, \tag{9.3}$$

since $B \otimes_A B$ is injectively contained in $B \overset{\circ}{*}_A B$ (Corollary 9.5). To prove the exactness of 9.3, we may tensorize the sequence with B over A and consider the exactness of

$$B \to B \otimes_A B \rightrightarrows B \otimes_A B \otimes_A B.$$

It is then easy to see that, as in commutative case, this sequence has actually splitting, and hence is exact (The reader may consult for example Proposition 2.18 of [10] for the precise arguments in commutative case). □

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