CONGRUENT ZETA FUNCTIONS. NO.09

YOSHIFUMI TSUCHIMOTO

It is interesting to regard things in the language of category theory. We give here a brief, incomplete, incorrect list. The entry in $\mathbb{F}_1$ are here just for fun.

		$\mathbb{F}_1$ math	$\mathbb{Z}$-modules	affine schemes
modules		(Sets)	(modules)	(sheaves)
algebras		(monoid)	((unital) ring) direct product tensor product	(affine scheme) disjoint union fiber product
morphisms		(map)	(ring hom) quotient map localization by an element	(morphism) closed immersion open immersion
			(Hopf algebra) (co-multiplication) (co-unit) (co-inverse)	(affine group scheme) (multiplication) (unit) (inverse)

9.1. **tensor products of modules over an algebra.** For those of you who are not familiar, we give a brief definition of tensor products.

Definition 9.1. Let A be a (not necessarily commutative) ring. Let M be a right A-module. Let N be a left A-module. Then we define the tensor product of M and N over A, denoted by

$$M \otimes_A N$$

as a module generated by symbols

$$\{m \otimes n; m \in M, n \in N\}$$

with the following relations.

(1)

$$(m_1 + m_2) \otimes n = m_1 \otimes n + m_2 \otimes n \quad (m_1, m_2 \in M,\ n \in N)$$

(2)

$$m \otimes (n_1 + n_2) = m \otimes n_1 + m \otimes n_2 \quad (m \in M,\ n_1, n_2 \in N)$$

(3)

$$ma \otimes n = m \otimes an \qquad (m \in M,\ n \in N,\ a \in A)$$

9.2. universality of tensor products.

Definition 9.2. Let A be a (not necessarily commutative) ring. Let M be a right A-module. Let N be a left A-module. Then for any module X, a map $f : M \times N \to X$ is said to be an A-balanced biadditive map if it satisfies the following conditions.

(1) $f(m_1 + m_2, n) = f(m_1, n) + f(m_2, n) \quad (\forall m_1, m_2 \in M, \forall n \in N)$
(2) $f(m, n_1 + n_2) = f(m, n_1) + f(m, n_2) \quad (\forall m \in M, \forall n_1, n_2 \in N)$
(3) $f(ma, n) = f(m, an) \quad (\forall m \in M, \forall n \in N, \forall a \in A)$

Lemma 9.3. *Let A be a (not necessarily commutative) ring. Let M be a right A-module. Let N be a left A-module. Then for any module X, there is a bijective additive correspondence between the following two objects.*

(1) *An A-balanced bilinear map $M \times N \to X$*
(2) *An additive map $M \otimes_A N \to X$*

Universality argmuments are deeply related to the uniqueness of initial objects. Consult Lang "Algebra".

9.3. additional structures on tensor products.

Lemma 9.4. *Let A, B be (not necessarily commutative) rings. Let M be a right A-module. Let N be a left A-module. If M carries a structure of a B-algebra (so that M is actually a B-A-bimodule,) then the tensor product $M \times_A N$ carries a structure of B-module in the following manner.*

$$b.(y \otimes n) = (xy) \otimes n \qquad (\forall b \in B, \forall y \in M, \forall n \in N)$$

Under the asummption of the lemma, we see that:

(1) $M \otimes_A N$ is additively generated by $\{m \otimes_A n | m \in M, n \in N\}$.
(2) $ma \otimes_A n = m \otimes_A an$ $(\forall m \in M, \forall n \in N, \forall a \in A)$
(3) $M \otimes_A N$ carries the structure of B-module.: $b.(m \otimes_A n) = (b.m) \otimes_A n \quad (\forall b \in B, \forall m \in M, \forall n \in N)$