

CONGRUENT ZETA FUNCTIONS. NO.5

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For any projective variety V over a field \mathbb{F}_q , we may define its congruent zeta function $Z(V/\mathbb{F}_q, T)$ likewise for the affine varieties.

We quote the famous Weil conjecture

CONJECTURE 5.1 (Now a theorem ¹). Let X be a projective smooth variety of dimension d . Then:

- (1) (Rationality) There exists polynomials $\{P_i\}$ such that

$$Z(X, T) = \frac{P_1(X, T)P_3(X, T) \dots P_{2d-1}(X, T)}{P_0(X, T)P_2(X, T) \dots P_{2d}(X, T)}.$$

- (2) (Integrality) $P_0(X, T) = 1 - T$, $P_{2d}(X, T) = 1 - q^dT$, and for each r , P_r is a polynomial in $\mathbb{Z}[T]$ which is factorized as

$$P_r(X, T) = \prod (1 - a_{r,i}T)$$

where $a_{r,i}$ are algebraic integers.

- (3) (Functional Equation)

$$Z(X, \frac{1}{q^dT}) = \pm q^{\frac{\chi}{2}} T^\chi Z(t)$$

where $\chi = (\Delta, \Delta)$ is an integer.

- (4) (Riemann Hypothesis) each $a_{r,i}$ and its conjugates have absolute value $q^{r/2}$.
- (5) If X is the specialization of a smooth projective variety X over a number field, then the degree of $P_r(X, T)$ is equal to the r -th Betti number of the complex manifold $X(\mathbb{C})$. (When this is the case, the number χ above is equal to the ‘‘Euler characteristic’’ $\chi = \sum_i (-1)^i b_i$ of $X(\mathbb{C})$.)

It is a profound theorem, relating the number of rational points $X(\mathbb{F}_q)$ of X over finite fields and the topology of $X(\mathbb{C})$.

For a further study we recommend [?, Appendix C], [?], [?].

projective space and projective varieties.

DEFINITION 5.2. Let R be a ring. A polynomial $f(X_0, X_1, \dots, X_n) \in R[X_0, X_1, \dots, X_n]$ is said to be **homogeneous** of degree d if an equality

$$f(\lambda X_0, \lambda X_1, \dots, \lambda X_n) = \lambda^d f(X_0, X_1, \dots, X_n)$$

holds as a polynomial in $n + 2$ variables $X_0, X_1, X_2, \dots, X_n, \lambda$.

DEFINITION 5.3. Let k be a field.

- (1) We put

$$\mathbb{P}^n(k) = (k^{n+1} \setminus \{0\})/k^\times$$

and call it (the set of k -valued points of) the **projective space**.

The class of an element (x_0, x_1, \dots, x_n) in $\mathbb{P}^n(k)$ is denoted by $[x_0 : x_1 : \dots : x_n]$.

¹There are a lot of people who contributed. See the references.

- (2) Let $f_1, f_2, \dots, f_l \in k[X_0, \dots, X_n]$ be homogenous polynomials. Then we set

$$V_h(f_1, \dots, f_l) = \{[x_0 : x_1 : x_2 : \dots : x_n]; f_j(x_0, x_1, x_2, \dots, x_n) = 0 \quad (j = 1, 2, 3, \dots, l)\}.$$

and call it (the set of k -valued point of) the **projective variety** defined by $\{f_1, f_2, \dots, f_l\}$.

(Note that the condition $f_j(x) = 0$ does not depend on the choice of the representative $x \in k^{n+1}$ of $[x] \in \mathbb{P}^n(k)$.)

LEMMA 5.4. *We have the following picture of \mathbb{P}^2 .*

(1)

$$\mathbb{P}^2 = \mathbb{A}^2 \coprod \mathbb{P}^1.$$

That means, \mathbb{P}^2 is divided into two pieces $\{Z \neq 0\} = \mathbb{C}V_h(Z)$ and $V_h(Z)$.

(2)

$$\mathbb{P}^2 = \mathbb{A}^2 \cup \mathbb{A}^2 \cup \mathbb{A}^2.$$

That means, \mathbb{P}^2 is covered by three “open sets” $\{Z \neq 0\}$, $\{Y \neq 0\}$, $\{X \neq 0\}$. Each of them is isomorphic to the plane (that is, the affine space of dimension 2).

5.1. proj.

DEFINITION 5.5. An \mathbb{N} -graded ring S is a commutative ring with a direct sum decomposition

$$S = \bigoplus_{i \in \mathbb{N}} S_i \quad (\text{as a module})$$

such that $S_i S_j \subset S_{i+j}$ ($\forall i, j \in \mathbb{N}$) holds. We define its irrelevant ideal S_+ as

$$S_+ = \bigoplus_{i>0} S_i.$$

An element f of S is said to be homogenous if it is an element of $\cup S_i$. An ideal of S is said to be homogeneous if it is generated by homogeneous elements. Homogeneous subalgebras are defined in a same way.

DEFINITION 5.6.

$$\text{Proj}(S) = \{\mathfrak{p}; \mathfrak{p} \text{ is a homogeneous prime ideal of } S, \mathfrak{p} \not\supset S_+\}$$

For any homogeneous element f of S , we define a subset D_f of $\text{Proj}(S)$ as

$$D_f = \{\mathfrak{p} \in \text{Proj } S; \mathfrak{p} \not\supset f\}.$$

$\text{Proj } S$ has a topology (Zariski topology) which is defined by employing $\{D(f)\}$ as an open base.

PROPOSITION 5.7. *For any graded ring S and its homogeneous element f , $S[\frac{1}{f}]$ also carries a structure of graded ring. There is a homeomorphism*

$$D_f \sim \text{Spec}(S[\frac{1}{f}]_0).$$

We may define, via these homeo altogether, a locally ringed space structure on $\text{Proj}(S)$.