Dolbeault complex of non-commutative projective varieties.

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Motivation

- To understand a symmetry of $H^{k,l} = H^l(X, \Omega^k)$

$$\bar{H^{k,l}} \cong H^{l,k}$$

  over fields of positive characteristics.
  - Deligne Illusie theory: $\bar{\partial}$ "resolution" of Ω^k is quasi isomorphic to the Frobenius "pullback" (somehow) of $\Omega^{k,l}$.
  - Cartier operators are in action.
- To obtain a lot of examples of non commutative objects.

Weyl algebras, Clifford algebras

$\Bbbk$: comutative field, $\operatorname{char} \Bbbk = p \gg 0, \operatorname{char} \Bbbk \neq 0$.
h, k, C: variables which commute with other variables ...

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Weyl algebra:

$$\mathrm{Weyl}_{n+1}^{(h,C)} = \Bbbk[h, C, X_0, X_1, \ldots, X_n, \bar{X}_0, \bar{X}_1, \ldots, \bar{X}_n]$$

relation (CCR): $[\bar{X}_i, X_j] = hC\delta_{ij}$.

Clifford algebra

$$\mathrm{Cliff}_{n+1}^{(h,C,k)} = \Bbbk[h, C, k, E_0, \ldots, E_n, \bar{E}_0, \ldots, \bar{E}_n]$$

relation(CAR): $[\bar{E}_i, E_j]_+ = Chk\delta_{ij}$.

Weyl-Clifford algebras

$$\mathrm{WC}_{n+1}^{(h,C,k)} = \mathrm{Weyl}_{n+1}^{(h,C)} \otimes_{\Bbbk[h,C]} \mathrm{Cliff}_{n+1}^{(h,C,k)}$$
$$= \Bbbk[h, C, k, X_0, \ldots, X_n, \bar{X}_0, \ldots, \bar{X}_n, E_0, \ldots, E_n, \bar{E}_0, \ldots, \bar{E}_n]$$

Existence of odd derivations $\partial, \bar{\partial}$:...

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Existence of odd derivations $\partial, \bar{\partial}$:

$$\partial : \begin{cases} X_i \mapsto E_i \\ \bar{X}_i \mapsto 0 \\ E_i \mapsto 0 \\ \bar{E}_i \mapsto k\bar{X}_i. \end{cases} \qquad \bar{\partial} : \begin{cases} X_i \mapsto 0 \\ \bar{X}_i \mapsto \bar{E}_i \\ E_i \mapsto -kX_i \\ \bar{E}_i \mapsto 0. \end{cases}$$

$E_i = \bar{\partial} X_i$, $\bar{E}_i = \bar{\partial} \bar{X}_i$.

things to note:

- $$\mathsf{WC}_{n+1} \cong \underbrace{\mathsf{WC}_1 \otimes \mathsf{WC}_1 \otimes \cdots \otimes \mathsf{WC}_1}_{n+1}$$

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- Logically by definition X and $\bar{X}$ are independent variables.

Presence of k

$$[\bar{E}_i, E_i]_+ = Chk$$

$$\boxed{[\bar{\partial}, \partial]_+ f = -k \,\mathrm{sdeg}_\mu(f) f.}$$

sdeg_μ :

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...For a plain $\mathbb{A}^{2n}$, k is not such a very good boy.

Super algebra structure of WC.

Before doing anything else, please keep in mind that we will use “super” notations. We define a signature of elements of WC:
$X_i, \bar{X}_i$: even.
$E_i, \bar{E}_i$:odd.
The symbol $[a, b]$ will be used to mean the super commutator instead of usual commutator.

$$[a, b] = ab - (-1)^{\hat{a}\cdot\hat{b}} ba$$

$\hat{a}, \hat{b}$: signature of a, b.

WC_1 (revisited)

$$WC_1 = \Bbbk[h, k, C, X, \bar{X}, E, \bar{E}]$$

$$[\bar{X}, X] = \bar{X}X - X\bar{X} = Ch$$

$$[\bar{E}, E] = \bar{E}E + E\bar{E} = Chk$$

$$E^2 = 0, \bar{E}^2 = 0$$

"X-variables" $(X, \bar{X})$ and "E-variables" $(E, \bar{E})$ commute:

$$[X, E] = 0, \quad [X, \bar{E}] = 0, \quad [\bar{X}, E] = 0, \quad [\bar{X}, \bar{E}] = 0.$$

WC_1 with the descritpion you would prefer

Let us denote $d = \partial + \bar{\partial}$: $E = dX, \bar{E} = d\bar{X}$.

$$WC_1 = \Bbbk[h, k, C, X, \bar{X}, dX, d\bar{X}]$$

$$[\bar{X}, X] = \bar{X}X - X\bar{X} = Ch$$

$$[\bar{d}X, dX] = Chk$$

$$(dX)^2 = 0, (d\bar{X})^2 = 0.$$

"X-variables" $(X, \bar{X})$ and "$d\bullet$-variables" $(dX, \bar{d}X)$ commute.
∂, $\bar{\partial}$ are computed in the same way as usual except:

$$\partial(d\bar{X}) = -kX, \quad \bar{\partial}(dX) = kX.$$

Shadow

- The Weyl algebra is a simple algebra when the base field $\mathbb{k}$ is of characteristic zero.
- When $\mathrm{char}(\mathbb{k}) \neq 0$ (as we always assume in this talk,) the Weyl algebra has a fairly large center.
- $\mathrm{Weyl}_{n+1}^{(h,C)}$ corresponds to a coherent sheaf of algebras on $\mathbb{A}^{n+1}_{\mathbb{k}[h,C]}$.
- We may obtain results over fields of characteristic 0 by using "ultra filters" on $\mathrm{Spm}(\mathbb{Z})$.

To do:

1. Construct a sheaf $\mathcal{A}$ of super algebras on $\mathbb{P}^n \times \mathbb{P}^n$.
2. See that $\mathcal{A}$ is a double complex with respect to $\partial, \bar{\partial}$.
3. $(\mathcal{A}, \bar{\partial})$ is quasi ismorphic to another sheaf of algebras on $\mathbb{P}^n \times \mathbb{P}^n$.
4. Computation of cohomollogy.
5. Mimic Deligne-Illusie theory.
6. Comarizon to the commutative theory by taking the limit $h \to 0$.
7. Watch $\partial \leftrightarrow \bar{\partial}$ symmetry.

A^{pre} (constraint with $\mu_R = 0$)

$$\mathrm{WC} = \Bbbk[h, C, k, X_0, \dots, X_n, \bar{X}_0, \dots, \bar{X}_n, E_0, \dots, E_n, \bar{E}_0, \dots, \bar{E}_n]$$

$$\mu_R = \sum_i (X_i \bar{X}_i k + E_i \bar{E}_i) - RkC$$

$$[\mu_R, f] = \mathrm{sdeg}_\mu(f) f.$$

$$(\mathrm{WC})_0 \stackrel{\mathrm{def}}{=} \{x \in WC; \mathrm{sdeg}_\mu(x) = 0\}$$

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$$\boxed{A^{\text{pre}} = (\mathrm{WC})_0 / (\mu_R)}$$

Marsden-Weinstein quotient.

Throw away torsions

$$A = \mathrm{Image}(A^{\mathrm{pre}} \to A^{\mathrm{pre}}[\frac{1}{k}]).$$

$$\mu_R = \sum_i (X_i \bar{X}_i k + E_i \bar{E}_i) - RkC = 0 \quad \text{in } A$$

$\Longrightarrow$

$$m := -\sum_i X_i \bar{X}_i = \frac{1}{k} \sum E_i \bar{E}_i \quad \text{in } A$$

$\Longrightarrow \; m(m - Ch)(m - 2Ch) \cdots (m - (n+1)Ch) = 0$ in A.
(Note that $(E_i \bar{E}_i)^2 = khE_i \bar{E}_i$ holds.)

(Secretly changed the sign of m compared to my november talk at MSJ.) (Oct.29: Secretly corrected the equation. We forgot to put some C's here.)

Dolbeault complex

1. We define the sheaf of super algebras $\mathcal{A}$ on $\mathbb{P}^n \times \mathbb{P}^n$ as the sheaf corresponding to A.
2. $\mathcal{A}$ is a double complex with respect to $\partial, \bar{\partial}$. (In particular,

$$[\bar{\partial}, \partial]_+ (= -k\,\mathrm{sdeg}_\mu) = 0.$$

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) Let us call it the **Dolbeault complex.**
3. We need to find a sheaf quasi isomorphic to $(\mathcal{A}, \bar{\partial})$.

Projective coordinate ring where $X_0 \neq 0$

$$
\begin{aligned}
A^{\heartsuit} &= A_{\{X_0 \neq 0\}} \\
&= \Bbbk[h, k, C, x_1, \ldots, x_n, x'_1, \ldots, x'_n, e_0, \ldots, e_n, e'_0, \ldots, e'_n, m]
\end{aligned}
$$

$$x_i = X_i X_0^{-1}, x'_i = X_0 \bar{X}_i, e_i = E_i X_0^{-1}, e'_i = X_0 \bar{E}_i$$

$$m = \frac{1}{k}(\sum_{i=0}^{n} e_i e'_i).$$

$$x_0 = 1, x'_0 = -\sum_{i=1}^{n} x_i x'_i - m$$

Ring structure of the projective coordinate ring where $X_0 \neq 0$

$$A^{\heartsuit} = \Bbbk[h, k, C, x_1, \ldots, x_n, x'_1, \ldots, x'_n, e_0, \ldots, e_n, e'_0, \ldots, e'_n, m]$$

$$[x'_i x_j] = hC\delta_{ij}$$
$$[x_i, x_j] = 0, [x'_i, x'_j] = 0$$
$$[e'_i, e_i](= [e'_i, e_i]_+) = Chk\delta_{ij}$$
$$e_i^2 = 0, (e'_i)^2 = 0$$
$$m = \frac{1}{k}\sum_{i=0}^{n} e_i e'_i \qquad (\text{ in } A^{\heartsuit}[\frac{1}{k}])$$

- In short, $A^{\heartsuit}$ is an algebra by adjoining e_0, e_0', m to the Weyl-Clifford algebra WC_n
- Essentially (probably up to "Morita equivalence"), we come back to our original WC_n.
- Note our covering $\cup_j \{X_j \neq 0\}$ of (non-commutative) $\mathbb{P}^n \times \mathbb{P}^n$ is only good for $\bar{\partial}$-action and is no good for ∂-action.

freeness of $A^\heartsuit$

- To concentrate on x, x', e, e'-variables, we denote
$$\Bbbk_3 = \Bbbk[h, C, k]$$
- $$A^\heartsuit \cong \Bbbk_3[x, x'] \otimes_{\Bbbk_3} \Bbbk_3[e, e', m]$$
- $\Bbbk_3[e, e', m]$ is a free finite module over $\Bbbk_3$

It follows that $A^\heartsuit$ corresponds to a finite free $\mathcal{O}$ module over $\mathbb{A}^n \times \mathbb{P}^n$

Freeness of $M = \Bbbk_3[e, e', m]$ (normal ordering)

(This slide is for the completeness sake only.)

- By using suitable commutation relations,

$$M = \sum \Bbbk_3 e^I m^{[I]} (e')^J$$
$$= \sum \Bbbk_3 e^I \frac{I!}{k^I} \sum_{|K|=I} e^K (e')^K (e')^J$$

The last module is isomorphic to a submodule M_1 of the exterior algebra

$$\Bbbk_3[\frac{1}{k}](\wedge(\oplus_{i=0}^n K e_i)) \otimes (\wedge(\oplus_{i=0}^n K e_i'))$$

M_1 is of the form $\Bbbk[h, k, C] \otimes_{\Bbbk[k]} M_0$ for some torsion free $\Bbbk[k]$-module M_0. By using a general theory of modules over PID, we see that M_0 is free. We may thus see that M is free.

Local quasi isomorphism

Theorem

$A^{\heartsuit}$ is quasi isomorphic to the following graded super subalgebra as a graded $\bar{\partial}$-complex.

$$\begin{aligned}\mathbb{k}[h, k, C, x_1, \dots, x_n, \beta_1, \dots, \beta_n, \\ (x'_1)^p, \dots, (x'_n)^p, (x'_1)^{p-1}e'_1, \dots, (x'_n)^{p-1}e'_n, \\ \epsilon - RCe_0]\end{aligned}$$

where

$$\beta_i = e_i - x_i e_0 \qquad (i = 1, 2, \dots, n)$$

$$\epsilon = \sum_{i=0}^{n} x'_i e_i,$$

explanation of variables(1)

$$
\begin{aligned}
\Bbbk[h, k, C, &\underline{x_1, \ldots, x_n, \beta_1, \ldots, \beta_n}, \\
&(x_1')^p, \ldots, (x_n')^p, (x_1')^{p-1} e_1', \ldots, (x_n')^{p-1} e_n', \\
&\epsilon - RCe_0]
\end{aligned}
$$

$$\beta_i = e_i - x_i e_0 = d(X_i/X_0)$$

One can think of $\Bbbk[h, k, C, x_1, \ldots, x_n, \beta_1, \ldots, \beta_n]$ as the ring of differentiable forms of (an affine piece of) $\mathbb{P}^n$.

explanation of variables(2)

$$\Bbbk[h, k, C, x_1, \ldots, x_n, \beta_1, \ldots, \beta_n, \underline{(x_1')^p, \ldots, (x_n')^p, (x_1')^{p-1}e_1', \ldots, (x_n')^{p-1}e_n'}, \epsilon - RCe_0]$$

One can think of

$$\Bbbk[h, k, C, (x_1')^p, \ldots, (x_n')^p, (x_1')^{p-1}e_1', \ldots, (x_n')^{p-1}e_n']$$

as the ring of differentiable forms of (an affine piece of) $\mathbb{P}^{n}$' twisted by Frob. Let us denote it by $\Omega_{\mathrm{sparse},\mathbb{P}^n}$.

Conclusion:

There exists a $\mathcal{O}_{\mathbb{P}^n}$-algebra $\mathcal{B}$ on $\mathbb{P}^n$ such that

1. $\mathcal{B}$ is an $\Omega^\bullet_{\mathbb{P}^n}$-algebra.
2. $\mathcal{B}$ is free of rank two as an $\Omega^\bullet_{\mathbb{P}^n}$-module.
3. $(\mathcal{A}, \bar{\partial})$ is quasi-isomorphic to $(\mathcal{B} \boxtimes \Omega_{\mathrm{sparse},\mathbb{P}^n}, 0)$.
4. $$\mathbb{R}\pi_{2*}\mathcal{A} \cong \mathcal{B} \boxtimes \bigoplus_j \mathbb{R}\Gamma(\mathbb{P}^n, \Omega^j)$$

what about varieties:

$V \subset \mathbb{P}^n$: algebraic variety
$\implies$ One can consider $\mathcal{A}/(I_V^p + \bar{I}_V^p)$.
This suggests some type of symmetry in cohomologies.