DOLBEAULT COMPLEXES OF NON-COMMUTATIVE PROJECTIVE VAIETIES.

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1. Weyl-Clifford algebras

In this section we define Weyl algebras, Clifford algebras and their tensor product which we call "Weyl Clifford algebras". The definition here is somewhat makeshift manner. We are not defining the most general object of what are called Weyl/Clifford algebras, but we take a special case with a special structure constatns. One more thing we need to be care about is that we use an extra variable C to homogenize the whole story. It will be then easy afterwards to consider "Proj" of our algebras to construct things that are proper("compact").

1.1. **The base field $\Bbbk$ and base ring $\Bbbk_1$.** In this paper, we fix a base field $\Bbbk$ and its extension commutative ring $\Bbbk_1$. We choose a specific element $h \in \Bbbk_1$. By adding an element h in this way, we may obtain a "commutative case" by specializing h to 0. In later sections $\Bbbk$ will be a field of characteristic $p \neq 0$ and $\Bbbk_1$ will be the ring $\Bbbk[h, \frac{1}{1-h^{p-1}}]$.

1.2. **Weyl algebras.** Weyl algebras play a role of function spaces on flat affine space. They are also studied extensively in the context of algebraic D-modules.

Definition 1.1. Let $\Bbbk, \Bbbk_1$ be a commutative rings as in subsection 1.1. Let h be an element of $\Bbbk_1$. The Weyl algebra is the following algebra.

$$\mathrm{Weyl}_{n+1}^{(h,C)} = \Bbbk_1[C, X_0, X_1, \dots, X_n, \bar{X}_0, \bar{X}_1, \dots, \bar{X}_n]$$

where $X_i, \bar{X}_j$ are subject to the following canonical commutation relations (CCR):

$$\begin{aligned} &[\bar{X}_i, X_j] = hC\delta_{ij} \quad \text{(Kronecker's delta)}, \\ &[\bar{X}_i, \bar{X}_i] = 0, \\ &[X_i X_j] = 0. \qquad (i, j = 0, 1, 2, \dots, n). \end{aligned}$$

C, h are both central.

C is a variable to homogenize the whole story.

When the base field $\mathbb{k}$ is of characteristic $p > 0$, we note that the following identity holds. It will be needed in later arguments.

$$(X_i\bar{X}_i)^p - (hC)^{p-1}X_i\bar{X}_i = X_i^p\bar{X}_i^p \qquad (i = 0, 1, 2, \dots, n). \tag{1.1}$$

1.3. **CAR(Clifford algebra).** We use Clifford algebras to study non-commutative analogue of "differential forms". (To be more accurate, since we are not using the most general form of Clifford algebras, we should refer to our algebras as "Clifford algebras of a special type".)

Definition 1.2. Let $\mathbb{k}, \mathbb{k}_1$ be a commutative rings as in subsection 1.1. Let h be an element of $\mathbb{k}_1$. We define the Clifford algebra as follows.

$$\mathrm{Cliff}_{n+1}^{(h,C,k)} = \mathbb{k}_1[C, k, E_0, \dots, E_n, \bar{E}_0, \dots, \bar{E}_n]$$

where the generators satisfy the following canonical anti-commutation relations(CAR):

$$[\bar{E}_i, E_j]_+ = Chk\delta_{ij}$$
$$[\bar{E}_i, \bar{E}_j]_+ = 0$$
$$[E_i, E_j]_+ = 0$$

Here again C, h, k are central elements.

Later we will use the elements E_i as analogue of differential 1-forms. The reader might then feel somewhat strange about the constant k. Later we will see that, when dealing with projective spaces, the variable k act as a constant (more appropriately, a $(1, 1)$-form) which represents the "curvature" of the space. In other words, it will be regarded as the "Kähler form" (which is the same thing as the curvature since $\mathbb{P}^n$ is Kähler-Einstein.)

Note that

$$(E_i\bar{E}_i)^2 = khCE_i\bar{E}_i$$

holds so that we have, when $\mathrm{char}(\mathbb{k}) = p > 0$,

$$(E_i\bar{E}_i)^p = (khC)^{p-1}E_i\bar{E}_i. \tag{1.2}$$

1.4. **Weyl-Clifford algebra.**

Definition 1.3. For any non negative integer n, m, we define Weyl-Clifford algebra as a tensor product

$$\mathrm{WC}_{n+1,m+1}^{(h,C,k)} = \mathrm{Weyl}_{n+1}^{(h,C)} \otimes_{\mathbb{k}_1[C]} \mathrm{Cliff}_{m+1}^{(h,C,k)}.$$

We are mainly interested in the case where $n = m$. In such a case we define

$$\mathrm{WC}_{n+1}^{(h,C,k)} = \mathrm{Weyl}_{n+1}^{(h,C)} \otimes_{\Bbbk_1[C]} \mathrm{Cliff}_{n+1}^{(h,C,k)}.$$

As a non-commutative analogue of "the space of differential form on $\mathbb{A}^{2(n+1)}$ with polynomial coefficients", we employ the Weyl-Clifford algebra $\mathrm{WC}_{n+1}^{(h,C,k)}$.

Note that the number n of even variables ($\{X_i,\ \bar{X}_i\}_{i=0}^{n}$) and the number m of even variables($\{E_i,\ E_i\}_{i=0}^{n}$) are equal. This may not be a "natural" assumption, but we employ the assumption to keep up with the ordinary theory of differential forms.

Later, when we deal with "non-commutative projective varieties". we will often encounter a situation where the number of odd variables ("forms") on our varieties may be bigger then the dimension of the variety.

1.5. **the degree and the signed degree.** We introduce the degree and the signed degree on WC. They are determined as follows:

$$\deg(X_i) = 1, \deg(\bar{X}_i) = 1, \deg(E_i) = 1, \deg(\bar{E}_i) = 1, \deg(C) = 2.$$

$$\mathrm{sdeg}(X_i) = 1, \mathrm{sdeg}(\bar{X}_i) = -1, \deg(E_i) = 1, \mathrm{sdeg}(\bar{E}_i) = -1, \mathrm{sdeg}(C) = 0.$$

1.6. GL**-action.** The Weyl Clifford algebra admits a GL-action. Namely, for any element $(g_{ij}) \in \mathrm{GL}_{n+1}(\Bbbk)$, we have

$$\begin{cases} X_i \mapsto \sum_j g_{ij} X_j \\ \bar{X}_k \mapsto \sum_l \breve{g}_{kl} X_l \\ E_i \mapsto \sum_j g_{ij} E_j \\ \bar{E}_k \mapsto \sum_l \breve{g}_{kl} E_l \end{cases}$$

where $(\breve{g}_{kl})$ is the transpose of the inverse of (g_{ij}):

$$\sum_j g_{ij} \breve{g}_{kj} = \delta_{ik}.$$

The Weyl Clifford algebra thus has a pretty large symmetry so that we may use it to study local picture of WC by studying specific open set. For example we will concentrate on an openset "$U^{\heartsuit}$" to see if our main result holds. We will further concentrate on "$x'_1 \neq 0$" to finish the proof of the main proposition.

2. SUPER COMMUTATOR AND SUPER ADJOINT

2.1. Form degree and the super algebra structure of WC. We define the form degree of elements of WC by

$$\mathrm{fdeg}(X_i) = 0, \mathrm{fdeg}(\bar{X}_i) = 0, \mathrm{fdeg}(E_i) = 1, \mathrm{fdeg}(\bar{E}_i) = -1,$$
$$\mathrm{fdeg}(k) = 2, \mathrm{fdeg}(h) = 0, \mathrm{fdeg}(C) = 0.$$

We employ the super algebra structure of the Weyl Clifford algebra WC defined in the preceding section by using fdeg as the super grading.

2.2. commutators and adjoints in super algebras. Before proceeding further, we note here a general notations and definitons about super algebras. Unlike other part of this paper, in this subsection we let A be an arbitrary super algebra. For any homogeneous elements f, g of the algebra A, we define their *super commutator* $[f, g]$ as

$$[f, g] \stackrel{\mathrm{def}}{=} fg - (-1)^{\hat{f}\hat{g}} gf$$

where $\hat{f}$, $\hat{g}$ are their super degree. We extend the super commutator linearly and define it for any pair of the algebra A [1].

For any element a of A, we define the *super adjoint* $\mathrm{ad}(a)$ as

$$\mathrm{ad}(a) : A \ni x \mapsto [a, x] \in A.$$

Please note onece again that we are taking super commutators.

3. Some important elements and operators.

In this section we introduce derivations $\partial, \bar{\partial}$ on WC_{n+1}. To do that, it is useful to introduce special elements ε and $\bar{\varepsilon}$. They are also important by themselves.

3.1. GL-invariant elements $\varepsilon, \bar{\varepsilon}$. The algebra WC has the following specific GL-invariant elements

$$\varepsilon = \sum_i \bar{X}_i E_i, \qquad \bar{\varepsilon} = \sum_i X_i \bar{E}_i$$

[1] When dealing with degrees, we often need to study first the case where elements are homogenous and then exntend it afterwards to the general case where elements are no longer homogenous. From this footnote on, when such occasion arise, we only deal with homogenous elements without mention. Extension to the general case should be easy.

3.2. $\partial, \bar{\partial}$ **as the** GL**-invariant derivations.** By a general theory (easy calculation) of super algebras, we know that the adjoint operators $\operatorname{ad}(\varepsilon), \operatorname{ad}\bar{\varepsilon}$ are odd derivation of the algebra WC. By calculating their action on generators of WC, we see that there exist odd derivations $\partial, \bar{\partial}$ on WC such that the relations

$$hC\partial = \operatorname{ad}\varepsilon, \qquad hC\bar{\partial} = -\operatorname{ad}\bar{\varepsilon}$$

hold. In other words, ∂ and $\bar{\partial}$ are the odd derivations on WC whose actions on the generators are summarized as follows:

$$\partial : \begin{cases} X_i \mapsto E_i \\ \bar{X}_i \mapsto 0 \\ E_i \mapsto 0 \\ \bar{E}_i \mapsto k\bar{X}_i. \end{cases} \qquad \bar{\partial} : \begin{cases} X_i \mapsto 0 \\ \bar{X}_i \mapsto \bar{E}_i \\ E_i \mapsto -kX_i \\ \bar{E}_i \mapsto 0. \end{cases}$$

Yes, we may use division by hC and write:

$$\partial = \frac{1}{hC}\operatorname{ad}(\varepsilon), \quad \bar{\partial} = \frac{1}{hC}\operatorname{ad}(\bar{\varepsilon}).$$

This is possible since WC is a free module over $\Bbbk_1[C]$.

We also note:

$$\bar{\partial}\varepsilon = -\mu_0. \tag{3.1}$$

For an element μ_0 defined in the next section.

3.3. **Normal ordering.** There are a lot of good account on normal orderings. The one the author learned on the subject is [2], which is very concise and mathematically clear.

In short, by employing certain order on variables, we may , by using commutation relations, choose a suitable basis on Weyl, Clifford algebras. For example, in our case the following proposition holds.

Proposition 3.1. WC_{n+1} *is a free module over* $\Bbbk_1$ *with the basis*

$$\{X^I \bar{X}^J E^K \bar{E}^L C^s k^t; I, J \subset \mathbb{N}^{n+1}, K, L \subset (\mathbb{N}_{<p})^{n+1}, s, t \in \mathbb{N}\}$$

where we put

$$\mathbb{N} = \mathbb{Z}_{\geq 0} = \{0, 1, 2, \dots\}.$$

4. The element μ_1 and our main algebra A

4.1. **Important assumptions.** From this section on, We assume $\operatorname{char}(\Bbbk) = p > 0$ and assume $h \notin \mathbb{F}_p^\times$. In other words, we will use

$$\Bbbk_1 = \Bbbk[h, \frac{1}{h^{p-1}}]$$

as a coefficient ring instead of $\Bbbk[h]$.

4.2. **The element μ_1.** For any $R \in \Bbbk$, let us consider the following specific element of WC.

$$\mu_R = k \sum_i X_i \bar{X}_i + \sum_i E_i \bar{E}_i - RkC.$$

It essentially gives a signed degree

$$\operatorname{ad}[\mu_R, f] = \operatorname{sdeg}(f) f \qquad (f \in \mathrm{WC}).$$

We would like to add a constraint $\mu_1 = 0$ on WC and obtain our target algebra A. Later we will see that in our algebra A, k is torsion free and may say that the equation

$$\sum_i X_i \bar{X}_i + (\frac{1}{k} \sum_i E_i \bar{E}_i) = C$$

holds in A. By using the equations (1.1) and (1.2), we see

$$\sum_i X_i^p \bar{X}_i^p = (1 - h^{p-1}) C^p \tag{4.1}$$

So we are considering the "sphere S^{n+1}".

Some readers may prefer considering more "natural" condition

$$\text{“} \sum_i X_i \bar{X}_i = 1 \text{”}$$

. But this condition, it seems, is not the right one to study for our purpose. For if we put the condition $\sum_i X_i \bar{X}_i = 1$ and assume also that our theory has good partial derivations ∂ and $\bar{\partial}$, we would have $\mu_0 = \partial(\sum_i X_i \bar{X}_i - 1) = 0$, which is stronger than our assumption.

We note also that we could have chosen $\mu_R = 0$ for a constant other than 1. It is easy to see that it is actually h/R that matters for the structure of A: by adjusting "scales", we may assume $R = 1$ as we do in this paper.

4.3. A **as the "Marsden-Weinstein quotient".** Since WC is a non-commutative ring, it is not a good idea to consider such thing as the quotient ring $\mathrm{WC}/(\mu_1)$ of WC by a both-sided ideal (μ_1) of WC generated by μ_1. If we try to see it in physics way, we need to drop off elements which are "canonical conjugate" to μ. If we try to see it in mathematics way, we need to consider the left ideal $J = \mathrm{WC} \cdot \mu_1$ and its idealizer

$$\mathbb{I}(J) = \{f \in WC; Jf \subset J\}.$$

We then consider the quotient algebra

$$\mathrm{MW}(\mathrm{WC}; J) \overset{\text{def}}{=} \mathbb{I}(J)/J.$$

A reader who is familiar with symplectic geometry may notice, by using the usual commutator-versus-poisson-bracket-correspondence, that the quotient $\mathrm{MW}(\mathrm{WC}; J)$ is a non commutative counterpart of the notion known as the Marsden-Weinstein quotient.

To study the object further, let us note that for any $f \in \mathrm{WC}$, we have

$$[\mu_1, f] = \mathrm{sdeg}(f) \tag{4.2}$$

where $\mathrm{sdeg}(f)$ is the signed degree of f, given by

$$\mathrm{sdeg}(X_i) = 1, \mathrm{sdeg}(\bar{X}_i) = -1, \mathrm{sdeg}(C) = 0.$$

When the base field $\Bbbk$ is a field of characteristic 0, the commutation relation (4.2) will yield the equation

$$\mathbb{I}(J) = (\mathrm{WC})_0 + \mathrm{WC}\,\mu_1,$$

so that we have the following description of the quotient.

$$\mathrm{MW}(\mathrm{WC}; J) = \mathbb{I}(J)/J = (\mathrm{WC})_{0/}(\mu_1). \tag{4.3}$$

When the characteristic of the base field $\Bbbk$ is equal to $p \neq 0$, The equation is not true any more, due to the fact that there are lots of elements with 0 derivatives, such as X_0^p.

To take advantage of parallelism between characteristics p and 0, we employ the right hand side of the equation (4.3) as a "better version" of the quotient, regardless of the characteristic of $\Bbbk$. Thus we define:

$$A^{\mathrm{pre}} = (\mathrm{WC})_0/(\mu_1).$$

A^{pre} defined above has a somewhat tricky part. It has k-torsions which are not easy to handle. Instead, we avoid such k-torsions

$$A = A^{\mathrm{pre}}/(k\text{-torsions}) = \mathrm{Image}(A^{\mathrm{pre}} \to A^{\mathrm{pre}}[\frac{1}{k}]).$$

5. THE CENTRAL SUBALGEBRA Z OF A

We define

$$Z = \Bbbk_1[\{X_i^p \bar{X}_j^p\}_{i,j=0}^n]$$

Proposition 5.1. $\mathbb{X} = \mathrm{Proj}(Z)$ *is isomorphic to the cartesian product* $\mathbb{P}^n \times \mathbb{P}^n$ *of projective spaces as a* $\Bbbk_1$*-scheme. A corresponds, (via the usual theory of correspondence between the graded* Z*-modules and the quasi coherent modules on* $\mathbb{X}$*), to a quasi coherent* $\mathcal{O}_{\mathbb{X}}$*-algebra.*

Proof. The reader may notice the resemblance of Z and the projective coordinate ring of the Segre embedding of $\mathbb{P}^n \times \mathbb{P}^n$. Using the "normal ordering" (Proposition 3.1) we see that there is no relation in $\{X_i^p \bar{X}_j^p\}$ other than the usual quadratic relations of Segre embedding.

□

Proposition 5.2. *We have:*

(1) *A is a graded algebra over the graded ring Z and so defines the associated quasi-coherent sheaf $\mathcal{A}$ on* $\mathrm{Proj}(Z) = \mathbb{P}^n \times \mathbb{P}^n$.
(2) *For any $n \in \mathbb{N}$, $\mathcal{A}/k^n\mathcal{A}$ is a coherent $\mathcal{O}_{\mathrm{Proj}(Z)}$-module.*

Proof. We only need to prove (2). As we have already mensioned in equation (4.1), we have

$$C^p = \frac{1}{1 - h^{p-1}} (\sum_i X_i^p \bar{X}_i^p).$$

The rest followis from (Proposition 3.1).

□

5.1. **A note on Proj.** In common textbook of (commutative) algebraic geometry (like [1]), projective schemes $\mathrm{Proj}(A)$ are considered for graded algebras A which are generated by the set A_1 of elements of degree one.

In this paper we need to consider a little bit different situation. Let us explain it here.

Let p be a positve integer. Let A,Z be graded algebras satisfying the following hypothesis:

(*)
A: $\mathbb{Z}$-graded algebra genearated by elements of degree 1.
Z: $p\mathbb{Z}$-graded algebra genearated by elements of degree p.
$\exists \{U_\lambda\}$: an open covering of $\mathrm{Proj}(Z)$, $\exists \{x_\lambda\} \subset A_1$ $\exists \{y_\lambda\} \subset A$ homogenious such that for each λ, $z_\lambda := x_\lambda y_\lambda$ is central in A and is invertible on U_λ.

Let us consider in this subsection the situation as in the hypothesis above. Let us note that, by "rescaling the grading" by $1/p$, $\mathrm{Proj}(Z)$ is defined in the usual way. But after the "rescaling", A (and usual $\mathbb{Z}$-graded A-modules M) are no longer $\mathbb{Z}$-graded. Let us consider (in the language of grading before the "rescaling",) $A_{p\mathbb{Z}} = \oplus_{k \in \mathbb{Z}} A_{pk}$ instead. We may then properly define sheaf $\mathcal{A}$ of algebra over $\mathrm{Proj}(Z)$ by putting

$$\mathcal{A} \stackrel{\mathrm{def}}{=} \widetilde{A_{p\mathbb{Z}}}$$

where the tilde sign which appears in the right hand side is the usual "sheafification" of $A_{p\mathbb{Z}}$. For any graded A-module M, the "associated sheaf" $\mathcal{M}$ on $\mathrm{Proj}(Z)$ is defined in the same way.

We of course need to remind ourselves about the modules $A_1, A_2, \dots A_{p-1}$ and $M_1, M_2, \dots M_{p-1}$. These things are regained by considering the Serre twist:

Definition 5.1. Under the hypothesis (*), we define a sheaf of modules $\mathcal{A}(n)$ for any integer n. It is a sheafification of a presheaf $\mathcal{A}(n)^{\mathrm{pre}}$ defined as follows:

$$\mathcal{A}(n)^{\mathrm{pre}}(U) = \left\{ f^{-1}a;\ \begin{matrix} f, a \in A(U)_{\text{homogeneous}}, \\ f: \text{ invertible on } U, \\ \deg(f) + n = \deg(a) \end{matrix} \right\}$$

For any graded A-module M, we define a presheaf $\mathcal{M}(n)^{\mathrm{pre}}$ as

$$\mathcal{M}(n)^{\mathrm{pre}}(U) = \left\{ f^{-1}m;\ \begin{matrix} m \in M_{\text{homogeneous}}, \\ f \in A_{\text{homogeneous}}, \\ f: \text{ invertible on } U, \\ \deg(f) + n = \deg(a) \end{matrix} \right\}$$

and define $\mathcal{M}(n)$ as the sheafification

Proposition 5.3. *Under the hypothesis (*), $\mathcal{A}(n)^{\mathrm{pre}}$ and $\mathcal{M}(n)^{\mathrm{pre}}$ are indeed presheaves on* $\mathrm{Proj}(Z)$*. Moreover, the following facts are true.*

(1) *If an element $x \in A_1$ is invertible on an open set U, then x is a generateing section on U of $\mathcal{A}(1)$ over $\mathcal{A}$-module.*
(2) *If an section $x \in A_1$ is invertible on an open set U, then for any $d \in \mathbb{Z}$ and for any $m \in M_d$, mx^{n-d} is a section of $\mathcal{M}(n)$.*
(3) *Let $\{U_\lambda\}$ and $\{z_\lambda\}$ as in the hypothesis (*). Then:*

$$M(n)(U_\lambda) = \{m z_\lambda^{-k}; m \in M, \deg(m) - k\deg(z_\lambda) = n\}.$$

In particular, it does not matter, when we conder $\mathcal{A}(n)$, whether we regard A as a left A-module (as we usually do in this paper) or a right A-module.
(4) *$\mathcal{A}(n)$ is a locally free left $\mathcal{A}$-module of rank one.*
(5) *$\mathcal{A}(n)$ is also a locally free right $\mathcal{A}$-module of rank one.*
(6) *$\mathcal{M}(n) \cong \mathcal{A}(n) \otimes_{\mathcal{A}} \mathcal{M}$.*
(7) *$\mathcal{A}(n) \otimes_{\mathcal{A}} \mathcal{A}(m) \cong \mathcal{A}(n+m)$.*

6. LOCAL TERMS

As we have described, the algebras we have constructed, such as WC , $(\mathrm{WC})_0$, A^{pre} and A, contains Z as a graded sub-algebra, and thus may be regarded as a sheaf of algebras over $\mathrm{Proj}(Z) = \mathbb{P}^n \times \mathbb{P}^n$. It would be important to use local coordinates and describe the situation locally and understand the algebras more clearly.

Let us consider an open set $U^{\heartsuit} = \{X_0^p \neq 0\} (\cong \mathbb{A}^n \times \mathbb{P}^n)$ of $\mathbb{P}^n \times \mathbb{P}^n$. For any graded module M over the graded ring Z, let us denote by $M^{\heartsuit}$ the "localization" of our objects to $U^{\heartsuit}$. To be more accurate, we consider the graded module

$$M^{\heartsuit} = \bigoplus_{s=0}^{\infty} \Gamma(U^{\heartsuit}, \tilde{M} \otimes (\mathcal{O}_{\mathbb{P}^n} \boxtimes \mathcal{O}_{\mathbb{P}^n}(s)))$$

of global sections of sheaf $\tilde{M}$ on $\mathrm{Proj}(Z)$ associated to M. Let us begin by the Weyl Clifford algebra:

$$\mathrm{WC}^{\heartsuit} = \mathrm{WC}[X_0^{-1}].$$

It has the 0-part:

$$(\mathrm{WC})_0^{\heartsuit} = \Bbbk_1[k, C, x_0, \dots, x_n, x'_0, \dots, x'_n, e_0, \dots, e_n, e'_0, \dots, e'_n]$$

where we put

$$x_i = X_i X_0^{-1}, \quad x'_i = X_0 \bar{X}_i, \quad e_i = E_i X_0^{-1}, \quad e'_i = X_0 \bar{E}_i.$$

Note that we have $x_0 = 1$ so we can drop it off. ***We need the two-stage conformation of the commutation relation.*** The generators which appear above satisfy the following CCR and CAR:

$$\boxed{\begin{array}{l} [x_i, x_j] = 0, \quad [x'_i, x'_j] = 0, \quad [x'_i, x_j] = hC\delta_{ij} \quad (i, j = 1, \dots, n) \\ [e_i, e_j]_+ = 0, [e'_i, e'_j]_+ = 0, [e'_i, e_j]_+ = Chk\delta_{ij} \quad (i, j = 0, 1, \dots, n). \end{array}}$$

Let us next consider the localzation $A^{\heartsuit}$ of our main object A. It is a quotient of $(\mathrm{WC})_0^{\heartsuit}$. The main relation is given by

$$\mu_1 = k \sum_i x_i x'_i + \sum e_i e'_i - kC = 0.$$

Since we deleted k-torsions, we may as well write:

$$\frac{1}{k} \sum_i e_i e'_i = C - \sum_i x_i x'_i.$$

Let us put the left hand side of the equation as m and rewrite the above equation as

$$x'_0 = C - \sum_{i=1}^{n} x_i x'_i - m.$$

Then by using this equation we may eliminate the variable x'_0 and obtain the following expression of $A^{\heartsuit}$.

$$A^{\heartsuit} = \Bbbk[k, h, C, x_1, \ldots, x_n, x'_1, \ldots, x'_n, e_0, \ldots, e_n, e'_0, \ldots, e'_n, m].$$
$$(m = \frac{1}{k}\sum_{i=0}^{n} e_i e'_i)$$

The generators which appear above satisfy the following CCR and CAR:

$$[x_i, x_j] = 0, \quad [x'_i, x'_j] = 0, \quad [x'_i, x_j] = hC\delta_{ij} \quad (i, j = 1, \ldots, n)$$
$$[e_i, e_j]_+ = 0, [e'_i, e'_j]_+ = 0, [e'_i, e_j]_+ = Chk\delta_{ij} \quad (i, j = 0, 1, \ldots, n).$$

In other words, $A^{\heartsuit}$ is an algebra obtained by adjoining an element m to an algebra $\Bbbk[k, h, C, x_1, \ldots, x_n, x'_1, \ldots, x'_n, e_0, \ldots, e_n, e'_0, \ldots, e'_n]$ which is isomorphic to the tensor product $\mathrm{Weyl}_n^{h,C} \otimes_{\Bbbk[h,C]} \mathrm{Cliff}_{n+1}^{h,C,k}$ of a Weyl algebra and a Clifford algebra. We note that this isomorphism preserves the 'anti holomorphic' derivation $\bar{\partial}$. and that it does not preserve the 'holomorphic' derivation ∂.

7. STRUCTURE OF $A^{\heartsuit}$

There exists an algebra $WC^{\heartsuit}_{(0)}$ such that the open set $U^{\heartsuit} = \{X_0 \neq 0\} \subset \mathrm{Proj}(WC_{(0)}$ is identified with $\mathrm{Proj}(\mathrm{WC}_{(0)})$.

$$\mathrm{WC}^{\heartsuit}_{(0)} = \Bbbk_1[C, k, \{x_i, x'_i, e_i, e'_i\}_{i=0}^{n}].$$

$$A^{\heartsuit} = \Bbbk_1[C, k, \{x_i, x'_i, e_i, e'_i\}_{i=0}^{n}, m].$$

$$m = \frac{1}{k}\sum_{i=0}^{n} e_i e'_i$$

Proposition 7.1.
$$A^{\heartsuit} = \Bbbk_1[C, k, \{x_j, x'_j\}_{j=1}^{n}, \{e_i, e'_i\}_{i=0}^{n}, m].$$

We define:
$$B = \Bbbk[h, k, C, \{x_j, x'_j, e_j, e'_j\}].$$
It is isomorphic to WC_n

「城崎」にあるように、 $A^{\heartsuit}$ の $\bar{\partial}$-cocycle は
$$\frac{1}{k}\bar{\partial}(e_0 b)$$
$e_0 b$: cocycle in $A^{\heartsuit}/kA^{\heartsuit}$. の形である。

$$A^{\heartsuit} = e_0 B[m] + e_0 B[m]\bar{\varepsilon} + B[m] + B[m]\bar{\varepsilon}$$

$$e_0 A^{\heartsuit} = e_0 B[m] + e_0 B[m]\bar{\varepsilon}$$

部分積分により、コサイクルは

$$\frac{1}{k}(e_0 f); \quad f \in B[m]$$

の形に直せる。m の次数を最小に取ろうとすると、m に関して最高次の係数を見ることにより、

$$\frac{1}{k}(e_0 f); \quad f \in B$$

の形のものに限って良いことがわかる。

$X_0\bar{X}_1 \neq 0$ で考えてみると、

$$b \in A^{\heartsuit}_{\text{sparse}}$$

の場合のみで良い。

$$e_0 B = e_0 R$$

R は半径の 2 乗の方ではなく、$dx, \bar{\partial}x'$ で生成される、いわば通常の form の空間。

DeRham, Dolbeault cohomology の関係に持ち込める。ただし、$\sum e_i e'_i = km$ の部分だけずれる。

$$\sum e_i e'_i = 0 \pmod{k}$$

で、 $\sum e_i e'_i$ は $\bar{\partial}\epsilon$ なので、 $\frac{1}{k}\bar{\partial}(e_0\epsilon)$ がでてくる。これが cocycles.

8. $\mathcal{A}_{\text{sparse}}$

Analogue of Deligne-Illusie theory.

$$(A, \bar{\partial}) \supset (A_{\text{sparse}}, 0)$$

quasi isomorphism

$$A_{\text{sparse}} \cong \exists S \boxtimes \Omega^{\bullet}_{\text{sparse}}$$

S is an extension:

$$0 \to \Omega^{\bullet} \to S \to \Omega^{\bullet} \to 0$$

whose extension class corresponds to the generator of $H^{1,1}$ via

$$H^{1,1} = H^1(\mathbb{P}^n, \Omega^1) \cong \operatorname{Ext}^1_{\mathbb{P}^n}(\mathcal{O}, \Omega^1) \to \operatorname{Ext}^1_{\mathbb{P}^n}(\Omega^{\bullet}, \Omega^{\bullet})$$

$$H^{\bullet}(A, \bar{\partial}) \cong \Bbbk[\eta] \otimes_{\Bbbk} \Bbbk[\varLambda]$$

関係式:

$$\eta^2 = 0, \qquad \varLambda^{n+1} = 0.$$

$\Bbbk[\varLambda]$ は $\mathbb{P}^n$ の cohomology 環.

9. $\mathcal{A}_{\mathrm{sparse},V}$

V:variety $\subset \mathbb{P}^n$ に対して、

$$A_V := A/(I_V^p, \bar{I}_V^p)$$

と定義する。

$$A_V \underset{q.i}{\sim} A_{\mathrm{sparse},V}$$

$$A_{\mathrm{sparse},V} \cong S_V \boxtimes \Omega_{V,\mathrm{sparse}}$$

$$R^i\Gamma(A_{\mathrm{sparse},V}, \bar{\partial}) \cong H(S_V) \otimes H(V, \Omega)$$

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