# $\mathbb{Z}_p$, $\mathbb{Q}_p$, AND THE RING OF WITT VECTORS

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No.03: $\boxed{\mathbb{Z}_p \text{ as a projective limit of } \{\mathbb{Z}/p^k\mathbb{Z}\}}$

DEFINITION 3.1. An ordered set $\Lambda$ is said to be **directed** if for all $i, j \in \Lambda$ there exists $k \in \Lambda$ such that $i \leq k$ and $j \leq k$.

DEFINITION 3.2. Let $\Lambda$ be a directed set. Let $\{X_\lambda\}_{\lambda \in \Lambda}$ be a family of topological rings. Assume we are given for each pair of elements $(\lambda, \mu) \in \Lambda^2$ such that $\lambda \geq \mu$, a continuous homomorphisms

$$\phi_{\mu\lambda} : X_\lambda \to X_\mu.$$

We say that such a system $(\{X_\lambda\}, \{\phi_{\mu\lambda}\})$ is a **projective system** of topological rings if it satisfies the following axioms.

(1) $\phi_{\nu\mu}\phi_{\mu\lambda} = \phi_{\nu\lambda} \quad (\forall\lambda, \forall\mu\forall\nu$ such that $\lambda \geq \mu \geq \nu)$.
(2) $\phi_{\lambda\lambda} = \mathrm{id} \quad (\forall\lambda \in \Lambda)$.

DEFINITION 3.3. Let $\mathcal{X} = (\{X_\lambda\}, \{\phi_{\mu\lambda}\})$ be a projective system of topological rings. Then we say that a **projective limit** $(X, \{\phi_\lambda\})$ of $\mathcal{X}$ is given if

(1) $X$ is a topological ring.
(2) $\phi_\lambda : X \to X_\lambda$ is a continuous homomorphism.
(3) $\phi_{\mu\lambda} \circ \phi_\lambda = \phi_\mu$ for $\forall\mu, \lambda$ such that $\lambda \geq \mu$.)
(4) $(X, \{\phi_\lambda\})$ is a universal object among objects which satisfy (1)-(3).

The "universal" here means the following: If $(Y, \psi_\lambda)$ satisfies

(1) $Y$ is a topological ring.
(2) $\psi_\lambda : Y \to X_\lambda$ is a continuous homomorphism.
(3) $\phi_{\mu\lambda} \circ \psi_\lambda = \psi_\mu$ for $\forall\mu, \lambda$ such that $\lambda \geq \mu$.)

Then there exists a unique continuous homomorphism

$$\Phi : Y \to X$$

such that

$$\psi_\lambda = \phi_\lambda \circ \Phi (\forall\lambda \in \Lambda).$$

PROPOSITION 3.4. *For any projective system of topological rings, a projective limit of the system exists. It is unique up to a unique isomorphism. (Hence we may call it* **the** *projective limit of the system.)*

PROOF. (sketchy) Put

$$X = \{(x_\lambda)_{\lambda\in\Lambda} \in \prod_{\lambda\in\Lambda} X_\lambda | \mathbb{P}\phi_{\mu,\lambda}(x_\lambda) = x_\mu \text{ for } \forall\mu, \forall\lambda \ \in \Lambda\} \subset \prod_{\lambda\in\Lambda} X_\lambda.$$

□

DEFINITION 3.5. For any projective system $(X, \{\phi_\lambda\})$ of topological rings, We denote the projective limit of it by

$$\varprojlim_{\lambda} X_\lambda.$$

Note: projective limits of systems of topological spaces, rings, groups, modules, and so on, are defined in a similar manner.

THEOREM 3.6.

$$\mathbb{Z}_p \cong \varprojlim_{k\to\infty} (\mathbb{Z}/p^k\mathbb{Z})$$

*as a topological ring.*

COROLLARY 3.7. *$\mathbb{Z}_p$ is a compact space.*

Note: There are several ways to prove the result of the above corollary. For example, the ring $\mathbb{Z}$ with the metric $d_p$ is easily shown to be totally bounded.

PROPOSITION 3.8. *Each element of $\mathbb{Z}_p$ is expressed uniquely as*

$$[0.a_1a_2a_3a_4\dots]_p \qquad (a_i \in \{0, 1, \dots, p-1\} \quad (i = 1, 2, 3, \dots)).$$

EXERCISE 3.1. Is $-4 = 1 - 5$ invertible in $\mathbb{Z}_5$? (Hint: use formal expansion

$$(1-x)^{-1} = 1 + x + x^2 + \dots$$

is it possible to write down a correct proof to see that the result is true?)