CONGRUENT ZETA FUNCTIONS. NO.5

YOSHIFUMI TSUCHIMOTO

For any projecvite variety V over a field $\mathbb{F}_q$, we may define its congruent zeta function $Z(V/\mathbb{F}_q, T)$ likewise for the affine varieties.

We quote the famous $\boxed{\text{Weil conjecture}}$

Conjecture 5.1 (Now a theorem [1]). Let X be a projective smooth variety of dimension d. Then:

(1) (Rationality) There exists polynomials $\{P_i\}$ such that

$$Z(X,T) = \frac{P_1(X,T)P_3(X,T)\dots P_{2d-1}(X,T)}{P_0(X,T)P_2(X,T)\dots P_{2d}(X,T)}.$$

(2) (Integrality) $P_0(X,T) = 1 - T$, $P_{2d}(X,T) = 1 - q^d T$, and for each r, P_r is a polynomial in $\mathbb{Z}[T]$ which is factorized as

$$P_r(X,T) = \prod (1 - a_{r,i}T)$$

where $a_{r,i}$ are algebraic integers.

(3) (Functional Equation)

$$Z(X, \frac{1}{q^d T}) = \pm q^{\frac{d\chi}{2}} T^\chi Z(t)$$

where $\chi = (\Delta.\Delta)$ is an integer.

(4) (Rieman Hypothesis) each $a_{r,i}$ and its conjugates have absolute value $q^{r/2}$.

(5) If X is the specialization of a smooth projective variety X over a number field, then the degeee of $P_r(X,T)$ is equal to the r-th Betti number of the complex manifold $X(\mathbb{C})$. (When this is the case, the number χ above is equal to the "Euler characteristic" $\chi = \sum_i (-1)^i b_i$ of $X(\mathbb{C})$.)

It is a profound theorem, relating the number of rational points $X(\mathbb{F}_q)$ of X over finite fields and the topology of $X(\mathbb{C})$.

For a further study we recommend [1, Appendix C],[2], [3].

References

[1] R. Hartshorne, *Algebraic geometry*, Springer Verlag, 1977.

[2] J. S. Milne, *Étale cohomology*, Princeton University Press, 1980.

[3] James S. Milne, *Lectures on etale cohomology (v2.21)*, 2013, Available at www.jmilne.org/math/, p. 202.

[1] There are a lot of people who contributed. See the references.