$\mathbb{Z}_p$, $\mathbb{Q}_p$, AND THE RING OF WITT VECTORS

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No.02: definition of $\mathbb{Z}_p$

Let p be a prime ("base"). We would like to introduce a metric on $\mathbb{Z}$ such that

$$n \text{ :small} \iff n \text{ is divisible by powers of } p.$$

Namely:

Definition 2.1. Let p be a prime number.

(1) We define a p-**adic norm** $|\bullet|_p$ on $\mathbb{Z}$ as follows.

$$|n|_p = \begin{cases} \frac{1}{p^k} & \text{if } n \neq 0 \text{ and } p^k | n \text{ and } p^{k+1} \not| n \\ 0 & \text{if } n = 0 \end{cases}$$

(2) We define a p-**adic distance** $d_p(\bullet, \bullet)$ on $\mathbb{Z}$ as follows.

$$d_p(n, m) = |n - m|_p \qquad (n, m \in \mathbb{Z})$$

Lemma 2.2. (1) *$|\bullet|_d$ enjoys the following properties.*

(a) $|x|_p \geq 0 \quad (\forall x \in \mathbb{Z}). \qquad |x|_p = 0 \iff x = 0.$

(b) $|x + y|_p \leq \max(|x|_p, |y|_p) \quad (\leq (|x|_p + |y|_p)$.

(c) $|xy|_p = |x|_p |y|_p$

(2) *$(\mathbb{Z}, d_p)$ is a metric space.*

Definition 2.3. A metric space (X, d) is said to be **complete** if every Cauchy sequence of X converges to an element of X.

Theorem 2.4. *Let (X, d) be a metric space. There exists a complete metric space $(\bar{X}, d)$ with an isometry $\iota : X \to \bar{X}$ such that X is dense in $\bar{X}$. Furthermore, $\bar{X}$ is unique up to a unique isometry.*

Definition 2.5. Let (X, d) be a metric space. We call $(\bar{X}, d)$ as in the above theorem **the completion** of (X, d).

Definition 2.6. Let p be a prime number. We denote the completion of $(\mathbb{Z}, d_p)$ by $(\mathbb{Z}_p, d_p)$ and call it **the ring of p-addic integers**. Thus elements of $\mathbb{Z}_p$ are **p-addic integers**.

Theorem 2.7. *$\mathbb{Z}_p$ has a unique structure of a topological ring. Namely,*

(1) *There exists unique continuous maps*

$$+ : \mathbb{Z}_p \times \mathbb{Z}_p \to \mathbb{Z}_p$$

(addition) and

$$\times : \mathbb{Z}_p \times \mathbb{Z}_p \to \mathbb{Z}_p$$

(multiplication) which are extensions of the usual addition and multiplication of $\mathbb{Z}$.

(2) *$(\mathbb{Z}_p, +, \times)$ is a commutative associative ring.*

DEFINITION 2.8. Let p be a prime number. For any sequence $\{a_j\}_{j=0}^{\infty}$ such that $a_j \in \{0, 1, 2, 3, \dots, p-1\}$, we consider a sequence $\{s_n\}$ defined by

$$s_n = \sum_{j=0}^{n} a_j p^j.$$

Then the sequence $\{s_n\}$ is a Cauchy sequence in $\mathbb{Z}_p$. We denote the limit of the sequence as

$$[0.a_0a_1a_2a_3\dots]_p.$$

EXERCISE 2.1. compute

$$[0.1]_3 + [0.2222\dots]_3$$