COHOMOLOGIES.

YOSHIFUMI TSUCHIMOTO

01.Review of elementary definitions on modules.

Definition 1.1. A (unital associative) **ring** is a set R equipped with two binary operations (addition ("+") and multiplication ("·")) such that the following axioms are satisfied.

(Ring1) R is an additive group with respect to the addition.
(Ring2) distributive law holds. Namely, we have

$$a(b+c) = ab + bc, \quad (a+b)c = ac + bc \qquad (\forall a, \forall b, \forall c \in R).$$

(Ring3) The multiplcation is associative.
(Ring4) R has a multiplicative unit.

For any ring R, we denote by 0_R (respectively, 1_R) the zero element of R (respectively, the unit element of R). Namely, 0_R and 1_R are elements of R characterized by the following rules.

- $a + 0_R = a, \quad 0_R + a = a \ \forall a \in R.$
- $a \cdot 1_R = a, \quad 1_R \cdot a = a \ \forall a \in R.$

When no confusion arises, we omit the subscript '$_R$' and write $0, 1$ instead of $0_R, 1_R$.

Definition 1.2. Let R be a unital associative ring. An R**-module** M is an additive group M with R-action

$$R \times M \to M$$

which satisfies

(Mod1) $(r_1 r_2).m = r_1.(r_2.m) \quad (\forall r_1, \forall r_2 \in R, \forall m \in M)$
(Mod2) $1.m = m \quad (\forall m \in M)$
(Mod3) $(r_1 + r_2).m = r_1.m + r_2.m \quad (\forall r_1, \forall r_2 \in R, \forall m \in M).$
(Mod4) $r.(m_1 + m_2) = r.m_1 + r.m_2 \quad (\forall r \in R, \forall m_1, \forall m_2 \in M).$

Example 1.3. Let us give some examples of R-modules.

(1) If k is a field, then the concepts "k-vector space" and "k-module" are identical.
(2) Every abelian group is a module over the ring of integers $\mathbb{Z}$ in a unique way.

Definition 1.4. An subset M of an R-module N is said to be an R**-submodule** of N if M itself is an R-module and the inclusion map $j : M \to N$ is an R-module homomorphism.

Definition 1.5. Let M, N be modules over a ring R. Then a map $f : M \to N$ is called an R**-module homomorphism** if it is additive and preserves the R-action.

The set of all module homomorphisms from M to N is denoted by $\mathrm{Hom}_R(M, N)$. It has an structure of an module in an obvious manner.

Definition 1.6. An subset N of an R-module M is said to be an R**-submodule** of M if N itself is an R-module and the inclusion map $j : N \to M$ is an R-module homomorphism.

DEFINITION 1.7. Let R be a ring. Let N be an R-submodule of an R-module M. Then we may define the **quotient** M/N by

$$M/N = M/\sim_N$$

where the equivalence relation $\sim_N$ is defined as follows:

$$m_1 \sim_N m_2 \quad \iff \quad m_1 - m_2 \in N.$$

It may be shown that the quotient M/N so defined is actually an R-module and that the natural projection

$$\pi : M \to M/N$$

is an R-module homomorphism.

DEFINITION 1.8. Let $f : M \to N$ be an R-module homomorphism between R-modules. Then we define its **kernel** as follows.

$$\operatorname{Ker}(f) = f^{-1}(0) = \{m \in M; f(m) = 0\}.$$

The kernel and the image of an R-module homomorphism f are R-modules.

THEOREM 1.9. *Let $f : M \to N$ be an R-module homomorphism between R-modules. Then*

$$M/\operatorname{Ker}(f) \cong f(N).$$

DEFINITION 1.10. Let R be a ring. An "sequence"

$$M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3$$

is said to be **an exact sequence of R-modules** if the following conditions are satisfied

(Exact1) M_1, M_2 are R-modules.
(Exact2) f, g are R-module homomorphisms.
(Exact3) $\operatorname{Ker}(g) = \operatorname{Image}(f)$.

For any R-submodule N of an R-module M, we have the following exact sequence.

$$0 \to N \to M \to M/N \to 0$$

EXERCISE 1.1. Compute the following modules.

(1) $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/3\mathbb{Z}, \mathbb{Z})$.
(2) $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z})$.
(3) $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}/5\mathbb{Z})$.

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$\boxed{\text{02. “Hom” modules.}}$

Lemma 2.1. *Let R be a ring. Let $f : M \to N$ be a homomorphism of R-modules. Then for any R-module L we may define:*

(1) *A homomorphism* $\mathrm{Hom}_R(L, f) : \mathrm{Hom}_R(L, M) \to \mathrm{Hom}_R(L, N)$ *defined by* $\mathrm{Hom}_R(L, f)(g) = f \circ g$.

(2) *A homomorphism* $\mathrm{Hom}_R(f, L) : \mathrm{Hom}_R(N, L) \to \mathrm{Hom}_R(M, L)$ *defined by* $\mathrm{Hom}_R(f, L)(h) = h \circ f$.

Proposition 2.2. *Let R be a ring. Let*

$$0 \to M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \to 0$$

be an exact sequence of R-modules. Then for any R-module N, we have:

(1)

$$0 \to \mathrm{Hom}_R(N, M_1) \overset{\mathrm{Hom}_R(N,f)}{\to} \mathrm{Hom}_R(N, M_2) \overset{\mathrm{Hom}_R(N,g)}{\to} \mathrm{Hom}_R(N, M_3)$$

is exact. The third arrow $\mathrm{Hom}_R(N, g)$ *need not be surjective.*

(2)

$$0 \to \mathrm{Hom}_R(M_3, N) \overset{\mathrm{Hom}_R(g,N)}{\to} \mathrm{Hom}_R(M_2, N) \overset{\mathrm{Hom}_R(f,N)}{\to} \mathrm{Hom}_R(M_1, N)$$

is exact. The third arrow $\mathrm{Hom}_R(f, N)$ *need not be surjective.*

Exercise 2.1. We consider an exact sequence

$$0 \to 3\mathbb{Z} \xrightarrow{i} \mathbb{Z} \to \mathbb{Z}/3\mathbb{Z} \to 0$$

where i is the inclusion map. Show that

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) \overset{\mathrm{Hom}(i,\mathbb{Z})}{\to} \mathrm{Hom}_{\mathbb{Z}}(3\mathbb{Z}, \mathbb{Z})$$

is not surjective

Exercise 2.2. Assume R is a field. Then show that the third arrow which appear in the sequence (1) in Proposition 2.2 is surjective.

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$\boxed{\text{03. cohomology of a complex.}}$ We mainly follow the treatment in [1].

Definition 3.1. Let R be a ring. A **cochain complex** of R-modules is a sequence of R-modules

$$C^\bullet : \dots \xrightarrow{d^{n-1}} C^n \xrightarrow{d^n} C^{n+1} \xrightarrow{d^{n+1}} \dots$$

such that $d^n \circ d^{n-1} = 0$. The n-th **cohomology** of the cochain complex is defined to be the R-module

$$H^n(C^\bullet) = \mathrm{Ker}(d^n)/\,\mathrm{Image}(d^{n-1}).$$

Elements of $\mathrm{Ker}(d^n)$ (respectively, $\mathrm{Image}(d^{n-1})$) are often referred to as **cocycles** (respectively, **coboundaries**).

A bit of category theory:

Definition 3.2. A **category** $\mathcal{C}$ is a collection of the following data

(1) A collection $\mathrm{Ob}(\mathcal{C})$ of **objects** of $\mathcal{C}$.
(2) For each pair of objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, a set
$$\mathrm{Hom}_{\mathcal{C}}(X, Y)$$
of **morphisms**.
(3) For each triple of objects $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$, a map("composition (rule)")
$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \times \mathrm{Hom}_{\mathcal{C}}(Y, Z) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$$

satisfying the following axioms

(1) $\mathrm{Hom}(X, Y) \cap \mathrm{Hom}(Z, W) = \emptyset$ unless $(X, Y) = (Z, W)$.
(2) (Existence of an identity) For any $X \in \mathrm{Ob}(\mathcal{C})$, there exists an element $\mathrm{id}_X \in \mathrm{Hom}(X, X)$ such that
$$\mathrm{id}_X \circ f = f, \quad g \circ \mathrm{id}_X = g$$
holds for any $f \in \mathrm{Hom}(S, X), g \in \mathrm{Hom}(X, T)$ ($\forall S, T \in \mathrm{Ob}(\mathcal{C})$).
(3) (Associativity) For any objects $X, Y, Z, W \in \mathrm{Ob}(\mathcal{C})$, and for any morphisms $f \in \mathrm{Hom}(X, Y), g \in \mathrm{Hom}(Y, Z), h \in \mathrm{Hom}(Z, W)$, we have
$$(f \circ g) \circ h = f \circ (g \circ h).$$

Morphisms are the basic actor/actoress in category theory.
An additive category is a category in which one may "add" some morphisms.

Definition 3.3. An additive category $\mathcal{C}$ is said to be **abelian** if it satisfies the following axioms.

(A4-1) Every morphism $f : X \to Y$ in $\mathcal{C}$ has a kernel $\ker(f) : \mathrm{Ker}(f) \to X$.
(A4-2) Every morphism $f : X \to Y$ in $\mathcal{C}$ has a cokernel $\mathrm{coker}(f) : Y \to \mathrm{Coker}(f)$.

(A4-3) For any given morphism $f : X \to Y$, we have a suitably defined isomorphism

$$l : \mathrm{Coker}(\ker(f)) \cong \mathrm{Ker}(\mathrm{coker}(f))$$

in $\mathcal{C}$. More precisely, l is a morphism which is defined by the following relations:

$$\ker(\mathrm{coker}(f)) \circ \overline{f} = f \ (\exists \overline{f}), \quad \overline{f} = l \circ \mathrm{coker}(\ker(f)).$$

Definition 3.4. Let $\mathcal{C}$ be an abelian category.

(1) An object I in $\mathcal{C}$ is said to be **injective** if it satisfies the following condition: For any morphism $f : M \to I$ and for any monic morphism $\iota : N \to M$, f "extends" to a morphism $\hat{f} : M \to I$.

$$\begin{tikzcd} M \arrow[r, "\hat{f}"] & I \\ N \arrow[u, "\iota"] \arrow[r, "f"] & I \arrow[u, equal] \end{tikzcd}$$

(2) An object P in $\mathcal{C}$ is said to be **projective** if it satisfies the following condition: For any morphism $f : P \to N$ and for any epic morphism $\pi : M \to N$, f "lifts" to a morphism $\hat{f} : M \to I$.

$$\begin{tikzcd} P \arrow[r, "\hat{f}"] \arrow[d, equal] & M \arrow[d, "\pi"] \\ P \arrow[r, "f"] & N \end{tikzcd}$$

Exercise 3.1. Let R be a ring. Let

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

be an exact sequence of R-modules. Assume furthermore that M_3 is projective. Then show that the sequence

$$0 \to \mathrm{Hom}_R(N, M_1) \overset{\mathrm{Hom}_R(N,f)}{\to} \mathrm{Hom}_R(N, M_2) \overset{\mathrm{Hom}_R(N,g)}{\to} \mathrm{Hom}_R(N, M_3) \to 0$$

is exact.

References

[1] S. Lang, *Algebra (graduate texts in mathematics)*, Springer Verlag, 2002.

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$\boxed{\text{04. projective and injective modules}}$

Lemma 4.1. *Let R be a (unital associative but not necessarily commutative) ring. Then for any R-module M, the following conditions are equivalent.*

1. *M is a direct summand of free modules.*
2. *M is projective*

Corollary 4.2. *For any ring R, the category* $(R\text{-modules})$ *of R-modules* **have enough projectives.** *That means, for any object $M \in (R\text{-modules})$, there exists a projective object P and a surjective morphism $f : P \to M$.*

Definition 4.3. Let R be a commutative ring. We assume R is a domain (that means, R has no zero-divisors except for 0.)

An R-module M is said to be **divisible** if for any $r \in R \setminus \{0\}$, the multplication map

$$M \stackrel{r\times}{\to} M$$

is surjective.

Definition 4.4. Let R be a commutative ring. We assume R is a domain (that means, R has no zero-divisors except for 0.)

An R-module M is said to be **divisible** if for any $r \in R \setminus \{0\}$, the multplication map

$$M \stackrel{r\times}{\to} M$$

is epic.

Lemma 4.5. *Let R be a (commutative) principal ideal domain (PID). Then an R-module I is injective if and only if it is divisible.*

Proposition 4.6. *For any (not necessarily commutative) ring R, the category* $(R\text{-modules})$ *of R-modules* **has enough injectives.** *That means, for any object $M \in (R\text{-modules})$, there exists an injective object I and an monic morphism $f : M \to I$.*

For the proof of the proposition above, we need the followin lemmas.

Lemma 4.7. *For any $\mathbb{Z}$-module M, let us denote by $\hat{M}$ the module $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{T}_1)$ where $T_1 = \mathbb{R}/\mathbb{Z}$. Then:*

1. *For any free $\mathbb{Z}$-module F, $\hat{F}$ is divisible (hence is $\mathbb{Z}$-injective).*
2. *For any $\mathbb{Z}$-module M, there is a canonical injective $\mathbb{Z}$-homomorphism $M \to \widehat{(\hat{M})}$.*
3. *Any Z-module M may be embeded in a divisible module T.*

Lemma 4.8. *Let T be a divisible module. Then for any ring A,*

$$\mathrm{Hom}_{\mathbb{Z}}(A, T)$$

is A-injective.

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05. projective and injective modules

DEFINITION 5.1. A (covariant) **functor** F from a category $\mathcal{C}$ to a category $\mathcal{D}$ consists of the following data:

(1) An function which assigns to each object C of $\mathcal{C}$ an object $F(C)$ of $\mathcal{D}$.
(2) An function which assigns to each morphism f of $\mathcal{C}$ an morphism $F(f)$ of $\mathcal{D}$.

The data must satisfy the following axioms:

(functor-1) $F(1_C) = 1_{F(C)}$ for any object C of $\mathcal{C}$.
(functor-2) $F(f \circ g) = F(f) \circ F(g)$ for any composable morphisms f, g of $\mathcal{C}$.

By employing the following axiom instead of the axiom (functor-2) above, we obtain a definition of a **contravariant functor**:
(functor-2′) $F(f \circ g) = F(g) \circ F(f)$ for any composable morphisms

DEFINITION 5.2. Let $F : \mathcal{C}_1 \to \mathcal{C}_2$ be a functor between additive categories. We call F **additive** if for any objects M, N in $\mathcal{C}_1$,

$$\operatorname{Hom}(M, N) \to \operatorname{Hom}(F(M), F(N))$$

is additive.

DEFINITION 5.3. Let F be an additive functor from an abelian category $\mathcal{C}_1$ to $\mathcal{C}_2$.

(1) F is said to be **left exact** (respectively, **right exact**) if for any exact sequence

$$0 \to L \to M \to N \to 0,$$

the corresponding map

$$0 \to F(L) \to F(M) \to F(N)$$

(respectively,

$$F(L) \to F(M) \to F(N) \to 0)$$

is exact
(2) F is said to be **exact** if it is both left exact and right exact.

LEMMA 5.4. *Let R be a (unital associative but not necessarily commutative) ring. Then for any R-module M, the following conditions are equivalent.*

(1) *M is a direct summand of free modules.*
(2) *M is projective*

COROLLARY 5.5. *For any ring R, the category* (R-modules) *of R-modules* **have enough projectives.** *That means, for any object $M \in$* (R-modules)*, there exists a projective object P and a surjective morphism $f : P \to M$.*

DEFINITION 5.6. Let R be a commutative ring. We assume R is a domain (that means, R has no zero-divisors except for 0.)

An R-module M is said to be **divisible** if for any $r \in R \setminus \{0\}$, the multplication map

$$M \stackrel{r\times}{\to} M$$

is surjective.

DEFINITION 5.7. Let R be a commutative ring. We assume R is a domain (that means, R has no zero-divisors except for 0.)

An R-module M is said to be **divisible** if for any $r \in R \setminus \{0\}$, the multplication map

$$M \stackrel{r\times}{\to} M$$

is epic.

DEFINITION 5.8. Let $(K^\bullet, d_K)$, $(L^\bullet, d_L)$ be complexes of objects of an additive category $\mathcal{C}$.

(1) A **morphism of complex** $u : K^\bullet \to L^\bullet$ is a family

$$u^j : K^j \to L^j$$

of morphisms in $\mathcal{C}$ such that u commutes with d. That means,

$$u^{j+1} \circ d_K^j = d_K^j \circ u^j$$

holds.

(2) A **homotopy** between two morphisms $u, v : K^\bullet \to L\bullet$ of complexes is a family of morphisms

$$h^j : K^j \to L^{j-1}$$

such that $u - v = d \circ h + h \circ d$ holds.

LEMMA 5.9. *Let $\mathcal{C}$ be an abelian category that has enough injectives. Then:*

(1) *For any object M in $\mathcal{C}$, there exists an* **injective resolution** *of M. That means, there exists an complex $I^\bullet$ and a morphism $\iota_M : M \to I^0$ such that*

$$H^j(I^\bullet) = \begin{cases} M \ (\text{via } \iota_M) & \text{if } j = 0 \\ 0 & \text{if } j \neq 0 \end{cases}$$

(2) *For any morphism $f : M \to N$ of $\mathcal{C}$, and for any injective resolutions $(I^\bullet, \iota_M)$, $(J^\bullet, \iota_N)$ of M and N (respectively), There exists a morphism $\bar{f} : I^\bullet \to J^\bullet$ of complexes which commutes with f. Forthermore, if there are two such morphisms $\bar{f}$ and f', then the two are homotopic.*

DEFINITION 5.10. Let $\mathcal{C}_1$ be an abelian category which has enough injectives. Let $F : \mathcal{C}_1 \to \mathcal{C}_2$ be a left exact functor to an abelian category. Then for any object M of $\mathcal{C}_1$ we take an injective resolution $I_M^\bullet$ of M and define

$$R^i F(M) = H^i(I_M^\bullet).$$

and call it the derived functor of F.

LEMMA 5.11. *The derived functor is indeed a functor.*

REFERENCES

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$\boxed{\text{06. Ext as a derived functor}}$

Let $\mathcal{C}$ be an abelian category. For any object M of $\mathcal{C}$, the extension group $\operatorname{Ext}^j_{\mathcal{C}}(M,N)$ is defined to be the derived functor of the "hom" functor

$$N \mapsto \operatorname{Hom}_{\mathcal{C}}(M,N).$$

We note that the Hom functor is a "bifunctor". We may thus consider the right derived functor of $\bullet \mapsto \operatorname{Hom}(\bullet, N)$ and that of $\bullet \mapsto \operatorname{Hom}(M,\bullet,N)$. Fortunately, both coincide: The extension group $\operatorname{Ext}^\bullet_{\mathcal{C}}(M,N)$ may be calculated by using either an injective resolution of the second variable N or a projective resoltuion of the first variable M. See [1, Proposition 8.4,Corollary 8.5].

EXAMPLE 6.1. Let us compute the extension groups $\operatorname{Ext}^j_{\mathbb{Z}}(\mathbb{Z}/36\mathbb{Z}, \mathbb{Z}/108\mathbb{Z})$.

(1) We may compute them by using an injective resolution

$$0 \to \mathbb{Z}/108\mathbb{Z} \to \mathbb{Q}/108\mathbb{Z} \to \mathbb{Q}/\mathbb{Z} \to 0$$

of $\mathbb{Z}/108\mathbb{Z}$.

(2) We may compute them by using a free resolution

$$0 \leftarrow \mathbb{Z}/36\mathbb{Z} \leftarrow \mathbb{Z} \leftarrow 36\mathbb{Z} \leftarrow 0$$

of $\mathbb{Z}/36\mathbb{Z}$.

EXERCISE 6.1. Compute an extension group $\operatorname{Ext}^j(M,N)$ for modules M,N of your choice. (Please choose a non-trivial example).

In the last lecture we mentioned the notion of injective hulls. Although they are not essential part of our lecture, students may find it interesting to calculate some of the injective hulls of known modules. So we write down some definitions and results related to them.

DEFINITION 6.2. Let M be an R-module. An R-module $E \supset M$ is called an **essential extension** of M if every non-zero submodule of E intersect M non-trivially. We denote this as $E \supset_e M$.

Such an essential extension is called maximal if no module properly containing E is an essential extension of M .

LEMMA 6.3. *A module M is injective if and only if M has no proper essential extensions.*

LEMMA 6.4. *Let R be a ring. Let $F \subset M$ be R-modules. We consider a family $\mathcal{F}$ of modules E which satisfy the following properties.*

- *E is an R-submodule of F which contains M.*
- *E is an essential extension of M.*

Then:

(1) *The set $\mathcal{F}$ has a maximal element.*
(2) *If F is an injective R-module, then any maximal element E of $\mathcal{F}$ is injective.*

THEOREM 6.5. *For any R-module M, there exists an injective module I which contains M whichis minimal among such. The module I is unique up to a (non-unique) isomorphism.*

DEFINITION 6.6. Such I in the above theorem is called the **injective hull** of M.

Injective hulls may then be used to obtain the "minimal injective resolution" of a module.

EXAMPLE 6.7. Let n be a positive integer. The injective hull of a $\mathbb{Z}$-module $\mathbb{Z}/n\mathbb{Z}$ is equal to $\mathbb{Z}[\frac{1}{n}]/n\mathbb{Z}$. Thus an injective resolution of $\mathbb{Z}/n\mathbb{Z}$ is given as follows.

$$0 \to \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}[\frac{1}{n}]/n\mathbb{Z} \to \mathbb{Z}[\frac{1}{n}]/\mathbb{Z} \to 0$$

REFERENCES

[1] S. Lang, *Algebra (graduate texts in mathematics)*, Springer Verlag, 2002.

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$\boxed{\text{07. Ext as a derived functor}}$ We recommend the book of Lang [1] as a good reference. The treatment here follows the book for the most part.

THEOREM 7.1. *Let $\mathcal{C}_1$ be an abelian category with enough injectives, and let $F : \mathcal{C}_1 \to \mathcal{C}_2$ be a covariant additive left functor to another abelian category $\mathcal{C}_2$. Then:*

(1) $F \cong R^0 F$.

(2) *For each short exact sequence*
$$0 \to M' \to M \to M'' \to 0$$
and for each $n \geq 0$ there is a natural homomorphism
$$\delta^n : R^n F(M'') \to R^{n+1} F(M)$$
such that we obtain a long exact sequence
$$\cdots \to R^n F(M') \to R^n F(M) \to R^n F(M'') \xrightarrow{\delta^n} R^{n+1} F(M') \to \dots.$$

(3) *δ is natural. That means, for a morphism of short exact sequences*
$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & N' & \longrightarrow & N & \longrightarrow & N'' & \longrightarrow & 0 \end{array}$$
the δ's give a commutative diagram:
$$\begin{array}{ccc} R^n F(M'') & \xrightarrow{\delta^n} & R^{n+1} F(M') \\ \downarrow & & \downarrow \\ R^n F(N'') & \xrightarrow{\delta^n} & R^{n+1} F(N') \end{array}$$

(4) *For each injective objective object I of A and for each $n > 0$ we have $R^n F(I)$.*

The collection $\{R^j F\}$ of functors $R^j F$ is a "universal delta functor". See [1].

LEMMA 7.2. *Under the assumption of the previous theorem, for any exact sequence $0 \to M' \to M \to M'' \to 0$ of objects in $\mathcal{C}_1$, there exists injective resolutions $I_{M'}, I_M, I_{M''}$ of M', M, M'' respectively and a commutative diagram*
$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I_{M'} & \longrightarrow & I_M & \longrightarrow & I_{M''} & \longrightarrow & 0 \end{array}$$

such that the diagram of resolutions is exact. Thus we obtain a diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & F(M') & \longrightarrow & F(M) & \longrightarrow & F(M'') & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & F(I_{M'}) & \xrightarrow{\alpha} & F(I_M) & \xrightarrow{\beta} & F(I_{M''}) & \longrightarrow & 0 \end{array}$$

such that each row in the last line is exact.

Note that j-th cohomology of the complex $F(I_M)$ (respectively, $F(I_{M'}), F(I_{M''})$) gives the $R^jF(M)(respectively, (R^jF(M'), R^jF(M'').)$ Using the resolution given in the lemma above, we may prove Theorem 7.1. Let us describe the map δ in more detail when $\mathcal{C}_2$ is a category of modules by "diagram chasing". Namely, for $x \in R^n(M'')$, let us show how to compute $\delta(x)$.

(1) $x \in R^n(M'')$ may be represented as a class $[c_x]$ of a cocycle $c_x \in \operatorname{Ker}(d : F(I^n_{M''}) \to F(I^{n+1}_{M''}))$.
(2) We take a "lift" $\tilde{c}_x \in F(I^n_M)$ such that $\beta^n(\tilde{c}_x) = c_x$. Note that $\tilde{c}_x$ is no longer a cocycle.
(3) Consider $e_x = d\tilde{c}_x \in F(I^{n+1}M)$. It is a coboundary and we have $\beta(e_x) = 0$.
(4) There thus exists an element $a_x \in F(I^n_{M'})$ such that $\alpha(a_x) = e_x$. a_x is no longer a coboundary but it is a cocycle.
(5) The cohomology class $[a_x]$ of a_x is the required $\delta(x)$.

Such computation appears frequently when we deal with cohomologies.

Definition 7.3. Let A be a ring. Let M, N be A-modules. Then an **extension** of N by M is a module L with a exact sequence

$$0 \to N \overset{\alpha}{\to} L \overset{\beta}{\to} M \to 0. \tag{E}$$

of A-modules. Let

$$0 \to N \overset{\alpha'}{\to} L' \overset{\beta'}{\to} M \to 0.$$

be another extension. Then the two extensions are said to be isomorphic if there exists a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \xrightarrow{\alpha} & L & \xrightarrow{\beta} & M & \longrightarrow & 0 \\ & & {\scriptstyle =}\downarrow & & \downarrow & & {\scriptstyle =}\downarrow & & \\ 0 & \longrightarrow & N & \xrightarrow{\alpha'} & L' & \xrightarrow{\beta'} & M & \longrightarrow & 0. \end{array}$$

Proposition 7.4. *There exists a bijection between the isomorphism classs of the extensions and elements of the* $\operatorname{Ext}^1_A(M,N)$. *The bijection is given by corresponding the extension (E) to the class* $\delta(1_N) \in \operatorname{Ext}^1(M,N)$ *of the identity map* 1_N *by* δ *associated to the exact sequence (E).*

See [1, XX,Exercise 27]

References

[1] S. Lang, *Algebra (graduate texts in mathematics)*, Springer Verlag, 2002.

COHOMOLOGIES.

YOSHIFUMI TSUCHIMOTO

08. Tensor products and Tor

Definition 8.1. Let A be an associative unital (but not necessarily commutative) ring. Let L be a right A-module. Let M be a left A-module. For any ($\mathbb{Z}$-)module N, an map

$$\varphi : L \times M \to N$$

is called an **A-balanced biadditive map** if

(1) $\varphi(x_1 + x_2, y) = \varphi(x_1, y) + \varphi(x_2, y) \quad (\forall x_1, \forall x_2 \in L, \forall y \in M)$.
(2) $\varphi(x, y_1 + y_2) = \varphi(x, y_1) + \varphi(x, y_2) \quad (\forall x \in L, \forall y_1, \forall y_2 \in M)$.
(3) $\varphi(xa, y) = \varphi(x, ay) \quad (\forall x \in L, \forall y \in M, \forall a \in A)$.

Proposition 8.2. *Let A be an associative unital (but not necessarily commutative) ring. Then for any right A-module L and for any left A-module M, there exists a ($\mathbb{Z}$-)module $X_{L,M}$ together with a A-balanced map*

$$\varphi_0 : L \times M \to X_{L,M}$$

which is universal amoung A-balanced maps.

Definition 8.3. We employ the assumption of the proposition above. By a standard argument on universal objects, we see that such object is unique up to a unique isomorphism. We call it the **tensor product** of L and M and denote it by

$$L \otimes_A M.$$

Lemma 8.4. *Let A be an associative unital ring. Then:*

(1) $A \otimes_A M \cong M$.
(2) $(L_1 \oplus L_2) \otimes_A M \cong (L_1 \otimes M) \oplus (L_2 \otimes_A M)$.
(3) *For any M, $L \mapsto L \otimes_A M$ is a right exact functor.*
(4) *For any right ideal J of A and for any A-module M, we have*

$$(A/J) \otimes_A M \cong M/J.M$$

In particular, if the ring A is commutative, then for any ideals I, J of A, we have

$$(A/I) \otimes_A (A/J) \cong A/(I + J)$$

Definition 8.5. For any left A-module M, the left derived functor $L_j F(M)$ of $F_M = \bullet \otimes_A M$ is called the Tor functor and denoted by $\mathrm{Tor}_j^A(\bullet, M)$.

By definition, $\mathrm{Tor}_j^A(L, M)$ may be computed by using projective resolutions of L.

Exercise 8.1. Compute $\mathrm{Tor}_j^{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z})$ for $n, m \in \mathbb{Z}_{>0}$.

References

COHOMOLOGIES.

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$\boxed{\text{09. Cohomology of groups}}$

Let G be a group. Let us consider a functor

$$F^G : M \mapsto M^G = \{m \in M; \quad g.m = m (\forall g \in G)\}$$

The functor is left-exact. The derived functor of this functor

$$H^j(G, M) = R^j F^G(M)$$

is called **the j-th cohomology of G with coefficients in** M. Let us consider $\mathbb{Z}$ as a G-module with trivial G-action. Then we may easily verify that

$$F^G(M) = M^G \cong \mathrm{Hom}_G(\mathbb{Z}, M).$$

Thus we have

$$H^j(G, M) = \mathrm{Ext}^j_G(\mathbb{Z}, M).$$

To compute cohomologies of G, it is useful to use $\mathbb{Z}[G]$-resolution of $\mathbb{Z}$. For any tuples $g_0, g_1, g_2, \dots, g_t$ of G, we introduce a symbol

$$[g_0, g_1, g_2, \dots, g_t]$$

and we consider the following sequence

$(*_G)$

$$0 \leftarrow \mathbb{Z} \xleftarrow{d} \bigoplus_{g_0 \in G} \mathbb{Z} \cdot [g_0] \xleftarrow{d} \bigoplus_{g_0, g_1 \in G} \mathbb{Z} \cdot [g_0, g_1] \xleftarrow{d} \bigoplus_{g_0, g_1, g_2 \in G} \mathbb{Z} \cdot [g_0, g_1, g_2] \xleftarrow{d} \dots$$

where ϵ, d are determined by the following rules.

$$\begin{aligned}
&d([g_0]) = 1 \\
&d([g_0, g_1]) = [g_1] - [g_0] \\
&d([g_0, g_1, g_2]) = [g_1, g_2] - [g_0, g_2] + [g_0, g_1] \\
&d([g_0, g_1, g_2, g_3]) = [g_1, g_2, g_3] - [g_0, g_2, g_3] + [g_0, g_1, g_3] - [g_0, g_1, g_2] \\
&\dots
\end{aligned}$$

To see that the sequence $*_G$ is acyclic, we consider a homotopy

$$h([g_0, g_1, \dots, g_t]) = [1, g_0, g_1, \dots, g_t]$$

Exercise 9.1. Show that $h \circ d + d \circ h = \mathrm{id}$

Lemma 9.1. (1) *Each of the modules that appears in the sequence $*_G$ admits an action of G determined by*

$$g.[g_0, g_1, g_2, \dots, g_t] = [g \cdot g_0, g \cdot g_1, g \cdot g_2, \dots, g \cdot g_t]$$

(2)

$$C_t = \bigoplus_{g_0, g_1, g_2, \dots g_t \in G} \mathbb{Z} \cdot [g_0, g_1, g_2, \dots, g_t]$$

is $\mathbb{Z}[G]$-free

There are several choices for the $\mathbb{Z}[G]$-basis of C_t. One such is clearly

$$\{[1, g_1, g_2, g_3, \dots, g_t]; g_1, g_2, \dots, g_t \in G\}.$$

It is traditional (and probably useful) to use another basis

$$\{\langle g_1, g_2, g_3, \dots, g_t\rangle; g_1, g_2, \dots, g_t \in G\}.$$

where

$$\langle g_1, g_2, g_3 \dots g_t\rangle = [1,\ g_1,\ g_1 g_2,\ g_1 g_2 g_3,\ \dots,\ g_1 g_2 g_3 \dots g_t].$$

Conversely we have

$$[1, a_1, a_2, \dots, a_t] = \langle a_1, a_1^{-1} a_2, a_2^{-1} a_3, \dots, a_{t-1}^{-1} a_t\rangle.$$

Definition 9.2. For any group G, the derived functor of a functor

$$F_G : (G - modules) \to (modules)$$

defined by

$$M \mapsto M_G = M/(\mathbb{Z} - \mathrm{span}\{g.m - m; g \in G, M \in M\})$$

is called the homology of G with coefficients in M. We denote the homology group $L_j F_G(M)$ by $H_j(G; M)$.

Lemma 9.3.

$$H_j(G; M) \cong \mathrm{Tor}_j^{\mathbb{Z}[G]}(\mathbb{Z}, M)$$

References

CATEGORIES, ABELIAN CATEGORIES AND COHOMOLOGIES.

YOSHIFUMI TSUCHIMOTO

Derived categories

We refer to [2],[1] for a good guide to the theory.
Main idea: Instead of dealing with an object of an additive category $\mathcal{C}$, we deal with complexes of $\mathcal{C}$. But:

(1) We want to regard quasi-isomorphic complexes as the "same".
(2) We want to identify two morphisms to be the same if they are homotopic.

11.1. **Cone of a complex.** Assume we are talking about complexes of objects in an additive category $\mathcal{C}$.

Definition 11.1. [2, 4.1] For any complex $X^\bullet$, we define $TX^\bullet$ to be a complex defined by

$$(TX)^i = X^{i+1}, \quad d_{TX} = -d_X.$$

Definition 11.2. [2, 4.3] Let $u : X^\bullet \to Y^\bullet$ be a morphism of complexes. **The cone** $C_u^\bullet$ of u is defined to be a graded object

$$Y^\bullet \oplus TX^\bullet$$

equipped with the following differential:

$$d \begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} d_Y & u \\ 0 & -d_X \end{pmatrix} \begin{pmatrix} y \\ x \end{pmatrix}$$

Idea 1: Instead of considering kernel and cokernel of a morphism u, we consider its cone C_u.

For any u, we have morphisms (triangle):

$$X^\bullet \xrightarrow{u} Y^\bullet \xrightarrow{\iota_Y} C_u^\bullet \xrightarrow{p_{TX}} TX^\bullet.$$

Let us call such a triangle **standard**. Now if $\mathcal{C}$ is abelian, then for each standard triangle as above we have the following long exact sequence:

$$\cdots \to H^k(X^\bullet) \to H^k(Y^\bullet) \to H^k(C_u^\bullet) \to H^{k+1}(X^\bullet) \to \dots$$

11.2. **The category $C(\mathcal{C})$.**

Definition 11.3. For any additive category $\mathcal{C}$, we define $C(\mathcal{C})$ to be The category of complexes of $\mathcal{C}$.

11.3. **The category $K(\mathcal{C})$.**

Definition 11.4. [2, 5.1] For any additive category $\mathcal{C}$, we define $K(\mathcal{C})$ to be

(1) $\mathrm{Ob}(K(\mathcal{C})) = \mathrm{Ob}(C(\mathcal{C}))$ (that means, objects of $K(\mathcal{C})$ are complexes).
(2) For any objects $X^\bullet, Y^\bullet$ of $K(\mathcal{C})$, we define

$$\mathrm{Hom}_{K(\mathcal{C})}(X^\bullet, Y^\bullet) = \mathrm{Hom}_{C(\mathcal{C})}(X^\bullet, Y^\bullet)/\mathrm{Homotopy}$$

Even if $\mathcal{C}$ is abelian, $K(\mathcal{C})$ is no longer abelian in general [2, 5.7]. But $K(\mathcal{C})$ has **distinguished triangles**, which are triangles isomorphic to standard triangles.

11.4. **The cateogory $D(\mathcal{C})$.** We assume $\mathcal{C}$ is an abelian category. We then add some inverses of quasi isomorphisms in $K(\mathcal{C})$ to define $D(\mathcal{C})$. $D(\mathcal{C})$ again is not necessarily be an abelian category, but it is a **triangulated category** which has distinguished triangles which satisfy certain axioms.

By considering only complexes which are bounded below, we may define $C^+(\mathcal{C}), K^+(\mathcal{C}), D^+(\mathcal{C})$ etc.

PROPOSITION 11.5. [2, 4.8] *If $\mathcal{C}$ has enoufh injectives then $D^+(\mathcal{C})$ is equivalent to $K^+(I(\mathcal{C}))$, where $I(\mathcal{C})$ is the category of injective objects in $\mathcal{C}$.*

So, in a sence, to consider an object $X^\bullet$ of $D^+(\mathcal{C})$ is to consider an injective resolution $I^\bullet$ of $X^\bullet$ and treat it up to homotopy.

For left-exact functor $\mathcal{C}_1 \to \mathcal{C}_2$, we may "define" (the actual definiton should be done more carefully. See [2])

$$\mathbb{R}F : D^+(\mathcal{C}_1) \to D^+(\mathcal{C}_2)$$

by

$$\mathbb{R}F(X^\bullet) = F(I^\bullet)$$

where $I^\bullet$ is an injective resolution of $X^\bullet$.

A good thing about treating derived functors in this way is that we may easily treat derived functors of compositions:

$$\mathbb{R}(F \circ G) \cong (\mathbb{R}F) \circ (\mathbb{R}G).$$

REFERENCES

[1] B. Iversen, *cohomology of sheaves*, Universitext in Mathematics, Springer Verlag, 1986.

[2] P.P.Grivel, *Catégorie dérivées et foncteurs dérivés*, In: Algebraic D-modules, Perspectives in mathematics **2** (1997), 1–108.