ALGEBRAIC GEOMETRY AND RING THEORY

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6.1. **sheaves.** Affine spectrum Spec(A) of a ring A carries one more important structure. Namely, its structure sheaf.

We will firstly review some definitions and first properties of sheaves. To illustrate the idea, we recall an easy lemma in topology.

Lemma 6.1 (Gluing lemma). *Let X, Y be a topological spaces. Let $\{U_\lambda\}_{\lambda\in\Lambda}$ be an open covering of X.*

(1) *If we are given a collection of continuous maps $\{f_\lambda : U_\lambda \to Y\}_{\lambda\in\Lambda}$ such that*
$$f_\lambda|_{U_\lambda\cap U_\mu} = f_\mu|_{U_\lambda\cap U_\mu}$$
holds for any pair $(\lambda,\mu)\in\Lambda^2$, then we have a unique continuous map $f : X \to Y$ such that
$$f|_{U_\lambda} = f_\lambda$$
holds for any $\lambda\in\Lambda$.

(2) *Conversely, if we are given a continuous map $f : X \to Y$, then we obtain a collection of maps $\{f_\lambda : U_\lambda \to Y\}_{\lambda\in\Lambda}$ by restriction.*

Proof. (1) It is easy to verify that we have a well-defined map
$$f : X \to Y$$
with
$$f|_{U_\lambda} = f_\lambda.$$
The continuity of f is proved by verifying that the inverse image of any open set $V \subset Y$ by f is open in X. □

6.1.1. *A convention.* Before proceeding further, we employ the following convention.

For an open covering $\{U_\lambda\}_{\lambda\in\Lambda}$ of a topological space X, we write
$$U_{\lambda\mu} = U_\lambda \cap U_\mu, \qquad U_{\lambda\mu\nu} = U_\lambda \cap U_\mu \cap U_\nu,$$
and so on.

6.1.2. *presheaves.* We first define presheaves.

Definition 6.2. Let X be a topological space. We say "a presheaf $\mathcal{F}$ of rings over X is given" if we are given the following data.

(1) For each open set $U \subset X$, a ring denoted by $\mathcal{F}(U)$. (which is called the ring of sections of $\mathcal{F}$ on U.)
(2) For each pair U, V of open subsets of X such that $V \subset U$, a ring homomorphism (called restriction)
$$\rho_{VU} : \mathcal{F}(U) \to \mathcal{F}(V).$$

with the properties

(1) $\mathcal{F}(\emptyset) = 0$.
(2) We have $\rho_{U,U} =$ identity for any open subset $U \subset X$.

(3) We have
$$\rho_{WV}\rho_{VU} = \rho_{WV}$$
for any open sets $U, V, W \subset X$ such that $W \subset V \subset U$.

6.1.3. *sheaves.*

DEFINITION 6.3. Let X be a topological space. A presheaf $\mathcal{F}$ of rings over X is called a sheaf if for any open set $U \subset X$ and for any open covering $\{U_\lambda\}_{\lambda\in\Lambda}$ of U, it satisfies the following conditions.

(1) ("Locality") If there is given a local section $f, g \in \mathcal{F}(U)$ such that
$$\rho_{U_\lambda U}(f) = \rho_{U_\lambda U}(g)$$
holds for all $\lambda \in \Lambda$, then we have $f = g$

(2) ("Gluing lemma"). If there is given a collection of sections $\{f_\lambda\}_{\lambda\in\Lambda}$ such that
$$\rho_{U_{\lambda\mu}U_\lambda}(f_\lambda) = \rho_{U_{\lambda\mu}U_\mu}(f_\mu)$$
holds for any pair $(\lambda, \mu) \in \Lambda^2$, then we have a section $f \in \mathcal{F}(U)$ such that
$$\rho_{U_\lambda U}(f) = f_\lambda$$
holds for all $\lambda \in \Lambda$.

We may similarly define sheaf of sets, sheaf of modules, etc.

LEMMA 6.4. *Let X be a topological set with an open base $\mathfrak{U}$. To define a sheaf $\mathcal{F}$ over X we only need to define $\mathcal{F}(U)$ for every member U of $\mathfrak{U}$ and check the sheaf axiom for open bases. In precise, given such data, we may always find a unique sheaf $\mathcal{G}$ on X such that $G(U) \cong F(U)$ holds in a natural way. (That means, the isomorphism commutes with restrictions wherever they are defined.)*

PROOF. Let $\mathcal{F}$ be such. For any open set $U \subset X$, we define a presheaf $\mathcal{G}$ by the following formula.
$$\mathcal{G}(U) = \left\{ (s_V) \in \prod_{V\in\mathfrak{U}, V\subset U} \mathcal{F}(V); \begin{array}{c} \rho_{WV}(s_V) = s_W \text{ for any } V, W \in \mathfrak{U} \\ \text{with the property } W \subset V \subset U. \end{array} \right\}$$
Restriction map of $\mathcal{G}$ is defined in an obvious manner.

Then it is easy to see that $\mathcal{G}$ satisfies the sheaf axiom and that
$$\mathcal{G}(U) \cong \mathcal{F}(U)$$
holds for any $U \in \mathfrak{U}$ in a natural way.

□

LEMMA 6.5. *Let A be a ring.*

(1) *We have a sheaf $\mathcal{O}$ of rings on* $\operatorname{Spec}(A)$ *which is defined uniquely by the property*
$$\mathcal{O}(O_f) = A_f \qquad (\forall f \in A)$$

(2) *For any A-module M we have a sheaf $\tilde{M}$ of modules on* $\operatorname{Spec}(A)$ *which is defined uniquely by the property*
$$\tilde{M}(O_f) = M_f \qquad (\forall f \in A)$$

(3) *For any A-module M, the sheaf $\tilde{M}$ is a sheaf of $\mathcal{O}$-modules on* $\operatorname{Spec}(A)$. *That means, it is a sheaf of modules over* $\operatorname{Spec}(A)$ *with an additional $\mathcal{O}$-action (which is defined in an obvious way.)*

PROOF. We prove (2).

From the previous Lemma, we only need to prove locality and gluing lemma for open sets of the form O_f. That means, in proving the properties (1) and (2) of Definition 6.3, we may assume that $U_\lambda = O_{f_\lambda}, U = O_f$ for some elements $f_\lambda, f \in A$.

Furthermore, in doing so we may use the identification $O_f \approx \operatorname{Spec} A_f$. By replacing A by A_f, this means that we may assume that $O_f = \operatorname{Spec}(A)$.

To sum up, we may assume

$$U = \operatorname{Spec}(A), U_\lambda = O_{f_\lambda}.$$

To simplify the notation, in the rest of the proof, we shall denote by

$$i_\lambda : M \to M_{f_\lambda}$$

the canonical map which we have formerly written i_{f_λ}. Furthermore, for any pair λ, μ of the index set, we shall denote by $i_{\lambda\mu}$ the canonical map

$$i_{\lambda\mu} M \to M_{f_\lambda f_\mu}.$$

Locality: Compactness of Spec(A) (Theorem **??**) implies that there exist finitely many open sets $\{O_{f_j}\}_{j=1}^k$ among U_λ such that $\cup_{j=1}^k O_{f_j} = \operatorname{Spec}(A)$. In particular there exit elements $\{c_j\}_{j=1}^k$ of A such that

$$c_1 f_1 + c_2 f_2 + \cdots + c_k f_k = 1 \tag{PU}$$

holds.

Let $m, n \in M$ be elements such that

$$i_j(m) = i_j(n) \qquad (\text{ in } M_{f_j}.)$$

With the help of the "module version" of Lemma **??**, we see that for each j, there exist positive integers N_j such that

$$f_j^{N_j}(m-n) = 0$$

holds for all $j \in \{1, 2, 3, \ldots, k\}$. Let us take the maximum N of $\{N_j\}$. It is easy to see that

$$f_j^N(m-n) = 0$$

holds for any j. On the other hand, taking (kN)-th power of the equation (PU) above, we may find elements $\{a_j\} \subset A$ such that

$$a_1 f_1^N + a_2 f_2^N + \cdots + a_k f_k^N = 1$$

holds. Then we compute

$$m - n = (a_1 f_1^N + a_2 f_2^N + \cdots + a_k f_k^N)(m-n) = 0$$

to conclude that $m = n$.

Gluing lemma:

Let $\{m_\lambda \in M_{f_\lambda}\}$ be given such that they satisfy

$$i_{\lambda\mu}(m_\lambda) = i_{\lambda\mu}(m_\mu)$$

for any λ, μ. We fist choose a finite subcovering $\{O_{f_j} = U_{\lambda_j}\}_{j=1}^k$ of $\{U_\lambda\}$. Then we may choose a positive integer N_1 such that

$$m_{\lambda_j} = x_j / f_j^{N_1} \qquad (\exists x_j \in M)$$

holds for all $j \in \{1, 2, 3, \ldots, k\}$.

$$i_{jl}(x_j f_l^{N_1}) = i_{jl}(x_l f_j^N)$$

Then by the same argument which appears in the "locality" part, there exists a positive integer N_2 such that

$$(f_i f_j)^{N_2}(x_j f_l^{N_1} - x_l f_j^{N_1}) = 0$$

holds for all $j, l \in \{1, 2, 3, \ldots, k\}$. We rewrite the above equation as follows.

$$(f_j^{N_2} x_j) f_l^{N_2+N_1} - (f_l^{N_2} x_l) f_j^{N_2+N_1} = 0.$$

On the other hand, by taking $k(N_1 + N_2)$-th power of the equation (PU), we may see that there exist elements $\{b_j\} \in A$ such that

$$\sum_{j=1}^{k} b_j f_j^{N_1+N_2} = 1$$

holds.

Now we put

$$n = \sum_j b_j (f_j^{N_2} x_j).$$

Then since for any l

$$(f_j^{N_2} x_j) = (f_l^{N_2} x_l) f_j^{N_2+N_1} / f_l^{N_2+N_1} = f_j^{N_2+N_1} m_{\lambda_l}$$

holds on O_l, we have $i_l(n) = m_{\lambda_l}$.

Now, take any other open set $O_{f_\mu} = U_\mu$ from the covering $\{U_\lambda\}$. $\{O_{f_j}\}_{j=1}^k \cup \{O_{f_\mu}\}$ is again a finite open covering of $\mathrm{Spec}(A)$. We thus know from the argument above that there exists an element n_1 of M such that

$$i_j(n_1) = m_{f_j}, \quad i_\mu(n_1) = m_\mu.$$

From the locality, n_1 coincides with n. In particular, $i_\mu(n) = m_\mu$ holds. This means n satisfies the requirement for the "glued object".

□

Corollary 6.6. *Let A be a commutative ring. Let B be a non-commutative ring which contains A as a central subalgebra (that means, $Z(B) \supset A$). Then there exists a sheaf $\tilde{B}$ of $\mathcal{O}$-algebras over* $\mathrm{Spec}(A)$

6.2. **Benefit of being a sheaf.** By saying that $\mathcal{O}$ is a sheaf on $\mathrm{Spec}(A)$, we may easily use the arguments we have used to proved the locality and the gluing lemma.

For example, the proof we gave in Theorem **??**, especially the part where we chose the idempotent p_1, was a bit complicated.

Let us give another proof using the sheaf arguments. There exists a unique element $p \in A = \mathcal{O}(\mathrm{Spec}(A))$ which coincides with 1 on $U_1 = V(J)$ and with 0 on $U_2 = V(I)$. From the uniqueness we see that

$$p^2 = p$$

holds since p^2 satisfies the same properties as p. The rest of the proof is the same.

As a second easier example, we consider the following undergraduate problem.

Problem: Find the inverse of the matrix

$$\begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}.$$

A student may compute (using "operations on rows") as follows.

$$\begin{pmatrix} 3 & 5 & | & 1 & 0 \\ 1 & 2 & | & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 5/3 & | & 1/3 & 0 \\ 1 & 2 & | & 0 & 1 \end{pmatrix}$$
$$\to \begin{pmatrix} 1 & 5/3 & | & 1/3 & 0 \\ 0 & 1/3 & | & -1/3 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 5/3 & | & 1/3 & 0 \\ 0 & 1 & | & -1 & 3 \end{pmatrix}$$
$$\to \begin{pmatrix} 1 & 0 & | & 2 & -5 \\ 0 & 1 & | & -1 & 3 \end{pmatrix}$$

The calculation is valid on $\mathrm{Spec}(\mathbb{Z}[1/3])$.

Another student may calculate (using "operations on columns") as follows.

$$\begin{pmatrix} 3 & 5 & | & 1 & 0 \\ 1 & 2 & | & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 3 & 5/2 & | & 1 & 0 \\ 1 & 1 & | & 0 & 1/2 \end{pmatrix}$$
$$\to \begin{pmatrix} 1/2 & 5/2 & | & 1 & 0 \\ 0 & 1 & | & -1/2 & 1/2 \end{pmatrix} \to \begin{pmatrix} 1 & 5/2 & | & 2 & 0 \\ 0 & 1 & | & -1 & 1/2 \end{pmatrix}$$
$$\to \begin{pmatrix} 1 & 1 & | & 2 & -5 \\ 0 & 1 & | & -1 & 3 \end{pmatrix}$$

The calculation is valid on $\mathrm{Spec}(\mathbb{Z}[1/2])$. Of course, both calculations are valid on the intersection $\mathrm{Spec}(\mathbb{Z}[1/2]) \cap \mathrm{Spec}(\mathbb{Z}[1/3]) = \mathrm{Spec}(\mathbb{Z}[1/6])$.

The gluing lemma asserts that the answer obtained individually is automatically an answer on the whole of $\mathrm{Spec}(\mathbb{Z})$. Of course, in this special case, there are lots of easier ways to tell that. But one may imagine this kind of thing is helpful when we deal with more complicated objects.

6.3. homomorphisms of (pre)sheaves.

DEFINITION 6.7. Let $\mathcal{F}_1$, $\mathcal{F}_2$ be presheaves of modules on a topological space X. Then we say that a sheaf homomorphism

$$\varphi : \mathcal{F}_1 \to \mathcal{F}_2$$

is given if we are given a module homomorphism

$$\varphi_U : \mathcal{F}_1(U) \to \mathcal{F}_2(U)$$

for each open set $U \subset X$ with the following property hold.

(1) For any open subsets $V, U \subset X$ such that $V \subset U$, we have

$$\rho_{V,U} \circ \varphi_U = \varphi_V \circ \rho_{V,U}.$$

(The property is also commonly referred to as "φ commutes with restrictions".)

DEFINITION 6.8. A homomorphism of sheaves is defined as a homomorphism of presheaves.

6.4. example of presheaves and sheafication. To proceed our theory further, we need to study a bit more about presheaves. Unfortunately, a sheaf of modules $\tilde{M}$ on an affine schemes are "too good". Namely, in terms of cohomology (which we study later,) we have always

$$H^i(\mathrm{Spec}(A), \tilde{M}) = 0 \qquad (\text{if } i > 0).$$

So to study some important problems on sheaf theory (which we will sure to encounter when we deal with non-affine schemes,) we need to study some examples from other mathematical areas.

A first example is a presheaf which satisfies the "locality" of sheaf axiom, but which fails to obey "gluing lemma".

EXAMPLE 6.9. Let $X = \mathbb{R}$ be the (usual) real line with the usual Lebesgue measure. Then we have a presheaf of L^1-functions given by

$$L^1(U) = \{f : U \to \mathbb{C}; |f| \text{ is integrable}\}.$$

L^1 is a presheaf which satisfies the "locality" of sheaf axiom, but which fails to obey "gluing lemma". Indeed, Let $\{U_n = (-n, n)\}$ be an open covering of $\mathbb{R}$ and define a section f_n on U_n by

$$f_n(x) = 1 \qquad (x \in U_n).$$

Then we see immediately that $\{f_n\}$ is a family of sections which satisfies the assumption of "gluing lemma". The function which should appear as the "glued function" is the constant function 1, which fails to be integrable on the whole of $\mathbb{R}$.

We may "sheaficate" the presheaf L^1 above. Instead of L^1-functions we consider functions which are locally L^1. Namely, for any open subset $U \subset \mathbb{R}$, we consider

$$L^1_{\mathrm{loc}}(U) = \left\{ f : U \to \mathbb{C}; \begin{array}{l} \forall x \in U, \exists V(\text{open in } U) \ni x \\ \text{such that } |f| \text{ is integrable on } V \end{array} \right\}$$

The presheaf so defined is a sheaf, which we may call "the sheaf of locally L^1-functions".

EXAMPLE 6.10. Similarly, we may consider a presheaf $U \mapsto \mathrm{Bdd}(U)$ of bounded functions on a topological space X. We may sheaficate this example and the sheaf so created is the sheaf of locally bounded functions.

EXAMPLE 6.11. It is psychologically a bit difficult to give an example of a presheaf which does not satisfy the locality axiom of a sheaf. But there are in fact a lot of them.

For any differentiable (C^∞) manifold X (students which are not familiar with the manifolds may take X as an open subset of $\mathbb{R}^n$ for an example.), we define a presheaf $\mathcal{G}$ on X defined as follows

$$\mathcal{G}(U) = C^\infty(U \times U) = \{\text{complex valued } C^\infty\text{-functions on } U \times U\}.$$

The restriction is defined in an obvious manner. It is an easy exercise to see that the presheaf does not satisfy the locality axiom of a sheaf.

To sheaficate this, we first need to introduce an equivalence relation on $\mathcal{G}(U)$.

$$f \sim g \iff \begin{pmatrix} \text{there exists an open covering } \{U_\lambda\} \text{ of } U \\ \text{such that } \rho_{U_\lambda, U} f = \rho_{U_\lambda, U} g \\ \text{for any } \lambda. \end{pmatrix}$$

Then we may easily see that

$$f \sim g \iff \begin{pmatrix} \text{there exists an open neighborhood } V \text{ of} \\ \text{the diagonal } \Delta_U \subset U \times U \\ \text{such that } f = g \text{ on } V \end{pmatrix}$$

holds.

Then we define

$$\mathcal{F}(U) = \mathcal{G}(U)/\sim .$$

It is now an easy exercise again to verify that $\mathcal{F}$ so defined is a sheaf. (Readers who are familiar with the theory of jets may notice that the sheaf is related to the sheaf of jets. In other words, there is a sheaf homomorphism from this sheaf to the sheaf of jets.)

6.5. **sheafication of a sheaf.** In the preceding subsection, we have not been explained what "sheafication" really means. Here is the definition.

LEMMA 6.12. *Let $\mathcal{G}$ be a presheaf on a topological space X. Then there exists a sheaf* $\mathrm{sheaf}(\mathcal{G})$ *and a presheaf morphism*

$$\iota_{\mathcal{G}} : \mathcal{G} \to \mathrm{sheaf}(\mathcal{G})$$

such that the following property holds.

(1) *If there is another sheaf $\mathcal{F}$ with a presheaf morphism*

$$\alpha : \mathcal{G} \to \mathcal{F},$$

then there exists a unique sheaf homomorphism

$$\tilde{\alpha} : \mathrm{sheaf}(\mathcal{G}) \to \mathcal{F}$$

such that

$$\alpha = \tilde{\alpha} \circ \iota_{\mathcal{G}}$$

holds.

Furthermore, such $\mathrm{sheaf}(\mathcal{G}), \iota_{\mathcal{G}}$ *is unique.*

DEFINITION 6.13. The sheaf $\mathrm{sheaf}(\mathcal{G})$ (together with $\iota_{\mathcal{G}}$) is called the sheafication of $\mathcal{G}$.

The proof of Lemma 6.12 is divided in steps.

The first step is to know the uniqueness of such sheafication. It is most easily done by using universality arguments. ([**?**] has a short explanation on this topic.)

Then we divide the sheafication process in two steps.

LEMMA 6.14. *(First step of sheafication) Let $\mathcal{G}$ be a presheaf on a topological space X. Then for each open set $U \subset X$, we may define a equivalence relation on $\mathcal{G}(U)$ by*

$$f \sim g \iff \left(\begin{array}{l} \textit{there exists an open covering } \{U_\lambda\} \textit{ of } U \\ \textit{such that } \rho_{U_\lambda,U} f = \rho_{U_\lambda,U} g \\ \textit{for any } \lambda. \end{array} \right)$$

Then we define

$$\mathcal{G}^{(1)}(U) = \mathcal{G}(U)/\sim .$$

Then $\mathcal{G}^{(1)}$ is a presheaf that satisfies the locality axiom of a sheaf. There is also a presheaf homomorphism from $\mathcal{G}$ to $\mathcal{G}^{(1)}$. Furthermore, $\mathcal{G}^{(1)}$ is universal among such.

LEMMA 6.15. *(Second step of sheafication) Let $\mathcal{G}$ be a presheaf on a topological space X which satisfies the locality axiom of a sheaf. Then we define a presheaf $\mathcal{G}^{(2)}$ in the following manner. First for any open covering $\{U_\lambda\}$ of an open set $U \subset X$, we define*

$$\mathcal{G}^{(2)}(U; \{U_\lambda\}) = \left\{ \{r_\lambda\} \in \prod_{\lambda \in \Lambda} G(U_\lambda);\ \begin{array}{l} \rho_{U_{\lambda\mu},U_\mu} f_\mu = \rho_{U_{\lambda\mu},U_\lambda} f_\lambda \\ \textit{for any } \lambda, \mu \in \Lambda. \end{array} \right\}$$

Then we define

$$\mathcal{G}^{(2)}(U) = \varinjlim_{\{U_\lambda\}} \mathcal{G}^{(2)}(U; \{U_\lambda\})$$

Then we may see that $G^{(2)}$ is a sheaf and that there exists a homomorphism from G to $G^{(2)}$. Furthermore, $G^{(2)}$ is universal among such.

Proofs of the above two lemma are routine work and are left to the reader.

Finish of the proof of Lemma 6.12: We put

$$\text{sheaf}(\mathcal{G}) = ((\mathcal{G})^{(1)})^{(2)}$$

□

6.6. **stalk of a presheaf.**

Definition 6.16. Let $\mathcal{G}$ be a presheaf on a topological space X. Let $P \in X$ be a point. We define the stalk of $\mathcal{G}$ on P as

$$\mathcal{G}_P = \varinjlim_{U \ni P} \mathcal{G}(U)$$

It should be noted at this stage that

Lemma 6.17. *Let $\mathcal{G}$ be a presheaf on a topological space X. The natural map*

$$\mathcal{G} \to \text{sheaf}(\mathcal{G})$$

induces an isomorphism of stalk at each point $x \in X$.

6.7. **kernels, cokernels, etc. on sheaves of modules.**

In this subsection we restrict ourselves to deal with sheaves of modules.

To shorten our statements, we call a presheaf which satisfies (only) the sheaf axiom (1) (locality) a "(1)-presheaf".

Lemma 6.18. *Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a homomorphism between sheaves of modules. Then we have*

(1) *The presheaf kernel of φ is a sheaf. We call it the sheaf kernel* $\operatorname{Ker}(\varphi)$ *of φ.*
(2) *The presheaf image of φ is not necessarily a sheaf, but it is a (1)-presheaf. We call the sheafication of the presheaf image as the sheaf image* $\operatorname{Image}(\varphi)$ *of φ.*
(3) *The presheaf cokernel of φ is not necessarily a sheaf. We call the sheafication of the cokernel as the sheaf cokernel* $\operatorname{Coker}(\varphi)$ *of φ.*

Definition 6.19. A sequence of homomorphisms of sheaves of modules

$$\mathcal{F}_1 \xrightarrow{f_1} \mathcal{F}_2 \xrightarrow{f_2} \mathcal{F}_3$$

is said to be exact if $\operatorname{Image}(f_1) = \operatorname{Ker}(f_2)$ holds.

Lemma 6.20. *A sequence of homomorphisms of sheaves of modules*

$$\mathcal{F}_1 \xrightarrow{f_1} \mathcal{F}_2 \xrightarrow{f_2} \mathcal{F}_3$$

is exact if and only if it is exact stalk wise, that means, if and only if the sequence

$$(\mathcal{F}_1)_P \xrightarrow{f_1} (\mathcal{F}_2)_P \xrightarrow{f_2} (\mathcal{F}_3)_P$$

is exact for all point P.