ALGEBRAIC GEOMETRY AND RING THEORY

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3.1. Examples of $\operatorname{Spec} A$.

(1) $\operatorname{Spec}(\mathbb{Z}) = \{(0)\} \cup \{(p) | p : \text{ prime }\}$.
(2) $\operatorname{Spec}(\Bbbk[X]) = \{(0)\} \cup \{(p) | p : \text{ irreducible polynomial}\}$.
(3) $\operatorname{Spec}(\mathbb{C}[X, Y]) = \{(0)\} \cup \{(p) | p : \text{ irreducible polynomial}\} \cup \{(X - a, Y - b) | a, b \in \mathbb{C}\}$.

3.2. Further properties of Spec.

LEMMA 3.1. *Let A be a ring. Then:*

(1) *For any $f \in A$, $D(f) = \{\mathfrak{p} \in \operatorname{Spec}(A); f \notin \mathfrak{p}\}$ is an open set of $\operatorname{Spec}(A)$.*
(2) *Given a point $\mathfrak{p}$ of $\operatorname{Spec}(A)$ and an open set U which contains $\mathfrak{p}$, we may always find an element $f \in A$ such that $\mathfrak{p} \in D(f) \subset U$. (In other words, $\{D(f)\}$ forms an open base of the Zariski topology.*

THEOREM 3.2. *For any ring A, $\operatorname{Spec}(A)$ is compact. (But it is not Hausdorff in most of the case.)*

DEFINITION 3.3. Let X be a topological space. A closed set F of X is said to be **reducible** if there exist closed sets F_1 and F_2 such that

$$F = F_1 \cup F_2, \quad F_1 \neq F, F_2 \neq F$$

holds. F is said to be **irreducible** if it is not reducible.

Recall that we have defined, for any ring A and for any ideal I, a closed subset $V(I)$ of $\operatorname{Spec}(A)$ by

$$V(I) = \{\mathfrak{p} \in \operatorname{Spec}(A); \operatorname{eval}_{\mathfrak{p}}(f) = 0 \quad (\forall f \in I).$$

We define:

DEFINITION 3.4. Let A be a ring. Let X be a subset of $\operatorname{Spec}(A)$. Then we define

$$I(X) = \{f \in A; \operatorname{eval}_{\mathfrak{p}}(f) = 0 \quad (\forall \mathfrak{p} \in I)\}.$$

LEMMA 3.5. *Let A be a ring. Then:*

(1) *For any subset X of $\operatorname{Spec}(A)$, $I(X)$ is an ideal of A.*
(2) *(For any subset S of A, $V(S)$ is a closed subset of $\operatorname{Spec}(A)$.)*
(3) *For any subsets $X_1 \subset X_2$ of $\operatorname{Spec}(A)$, we have $I(X_1) \supset I(X_2)$.*
(4) *For any subsets $S_1 \subset S_2$ of A, we have $V(S_1) \supset V(S_2)$.*
(5) *For any subset X of $\operatorname{Spec}(A)$, we have $V(I(X)) \subset X$.*
(6) *For any subset S of A, we have $I(V(S)) \subset S$.*

COROLLARY 3.6. *Let A be a ring. Then:*

(1) *For any subset X of $\operatorname{Spec}(A)$, we have $I(V(I(X))) = I(X)$.*
(2) *For any subset S of A, we have $V(I(V(S))) = V(S)$.*

DEFINITION 3.7. Let I be an ideal of a ring A. Then we define its **radical** to be

$$\sqrt{I} = \{x \in A; \exists N \in \mathbb{Z}_{>0} \text{ such that } x^N \in I\}.$$

PROPOSITION 3.8. *Let A be a ring. Then;*

(1) *For any ideal I of A, we have $V(I) = V(\sqrt{I})$.*
(2) *For two ideals I, J of A, $V(I) = V(J)$ holds if and only if $\sqrt{I} = \sqrt{J}$.*
(3) *For an ideal I of A, $V(I)$ is irreducible if and only if $\sqrt{I}$ is a prime ideal.*