ALGEBRAIC GEOMETRY AND RING THEORY

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2.1. Spec A.

DEFINITION 2.1. An ideal I of a ring A is said to be

(1) a prime ideal if A/I is an integral domain.
(2) a maximal ideal if A/I is a field.

DEFINITION 2.2. Let A be a ring. Then we define its *affine spectrum* as

$$\operatorname{Spec}(A) = \{\mathfrak{p} \subset A; \mathfrak{p} \text{ is a prime ideal of } A\}.$$

DEFINITION 2.3. Let A be a ring. For any $\mathfrak{p} \in \operatorname{Spec}(A)$ we define "evaluation map" $\operatorname{eval}_\mathfrak{p}$ as follows:

$$\operatorname{eval}_\mathfrak{p} : A \to A/\mathfrak{p}$$

Note that $A/\mathfrak{p}$ is a subring of a field $Q(A/\mathfrak{p})$, the field of fractions of the integral domain $A/\mathfrak{p}$. We interpret each element f of A as a something of a "fuction", whose value at a point $\mathfrak{p}$ is given by $\operatorname{eval}_\mathfrak{p}(f)$.

We introduce a topology on $\operatorname{Spec}(A)$. We basically mimic the following Lemma:

LEMMA 2.4. *Let X be a topological space. then for any continuous function $f : X \to \mathbb{C}$, its zero points $\{x \in X; f(x) = 0\}$ is a closed subset of X. Furthermore, for any family $\{f_\lambda\}$ of continous $\mathbb{C}$-valued functions, its common zeros $\{x \in X; f_\lambda(x) = 0 \quad (\forall \lambda)\}$ is a closed subset of X.*

DEFINITION 2.5. Let A be a ring. Let S be a subset of A, then we define the common zero of S as

$$V(S) = \{\mathfrak{p} \in \operatorname{Spec}(A); \operatorname{eval}_\mathfrak{p}(f) = 0 \qquad (\forall f \in S\}.$$

For any subset S of A, let us denote by $\langle S \rangle_A$ the ideal of A generated by S. Then we may soon see that we have $V(S) = V(\langle S \rangle_A)$. So when thinking of $V(S)$ we may in most cases assume that S is an ideal of A.

LEMMA 2.6. *Let A be a ring. Then:*

(1) $V(0) = \operatorname{Spec}(A), \quad V(\{1\})(= V(A)) = \emptyset$.
(2) *For any family $\{I_\lambda\}$ of ideals of A, we have $\cap_\lambda V(I_\lambda) = V(\sum_\lambda I_\lambda)$.*
(3) *For any ideals I, J of A, we have $V(I) \cup V(J) = V(I \cdot J)$.*

PROPOSITION 2.7. *Let A be a ring. $\{V(I); I$ is an ideal of $A\}$ satisfies the axiom of closed sets of $\operatorname{Spec}(A)$. We call this the Zariski topology of $\operatorname{Spec}(A)$.*

PROBLEM 2.8. Prove Lemma 2.6.