ALGEBRAIC GEOMETRY AND RING THEORY

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1.1. **Introduction.** For any topological space X, we define

$$C(X) = \{X \to \mathbb{C};\ \text{continuous}\}.$$

It has a natural structure of a ring by introducing "point-wise operations":

$$(f+g)(x) = f(x)+g(x), \quad (f\cdot g)(x) = f(x)\cdot g(x) \quad (\forall x \in X, \forall f, g \in C(X)).$$

It has an extra structure of $*$-operation:

$$(f^*)(x) = \overline{f(x)} \qquad (complexconjugate).$$

and a topology (locally uniform topology) which we shall not describe in detail.

THEOREM 1.1 (Gelfand-Naimark). *("Commutative Case")*

$$\text{(Compact Hausdorff space)} \ni K \mapsto C(K) \in (C^*\text{-algebras})$$

is a bijection.

The inverse of the correspondence above is given by associating to a commutative C^*-algebra A a set

$$\mathrm{Spm}(A) = \{\text{maximal ideal of } A\}$$

with a certain topology.

A first interesting part of modern algebraic geometry is that we may mimic the correspondence in the Gelfand-Naimark theorem above and associate to any commutive ring a compact (but not Hausdorff) space $\mathrm{Spec}(A)$. The elements of A may then be considered as "continuous functions" on $\mathrm{Spec}(A)$.

The upshot is that we may "cut and paste", as one usually does with functions, elements of abstract commutative rings. Any other method of functional analysis also has the possibility to be applied in the commutative ring theory.

On the other hand, it is possible to manipulate the compact space $\mathrm{Spec}(A)$ and create new algebras out of the existing commutative ring A. We may furtheremore paste such $\mathrm{Spec}(A)$'s altogther and define another geometric objects.

PROBLEM 1.2. Let X be a finite set with the discrete topology. Show that $C(X)$ has exactly $\#X$ pieces of maximal ideals.