$\mathbb{Z}_p$, $\mathbb{Q}_p$, AND THE RING OF WITT VECTORS

No.06: $\boxed{\mathbb{Q}_p}$

DEFINITION 6.1. We denote by $\mathbb{Q}_p$ the quotient field of $\mathbb{Z}_p$.

LEMMA 6.2. *Every non zero element $x \in \mathbb{Q}_p$ is uniquely expressed as*

$$x = p^k u \qquad (k \in \mathbb{Z}, u \in \mathbb{Q}_p^\times).$$

We have so far constructed a ring $\mathbb{Z}_p$ and a field $\mathbb{Q}_p$ for each prime p.

PROPOSITION 6.3. *Let p be a prime. Then:*

(1) *$\mathbb{Z}_p$ is a local ring with the unique maximal ideal $p\mathbb{Z}_p$.*

(2)
$$\mathbb{Z}_p/p\mathbb{Z}_p \cong \mathbb{F}_p(= \mathbb{Z}/p\mathbb{Z}).$$

(3) *$\mathbb{Z}_p$ is an integral domain whose quotient field $\mathbb{Q}_p$ is a field of characteristic zero.*

With $\mathbb{Q}_p$ and/or $\mathbb{Z}_p$, we may do some "calculus" such as:

THEOREM 6.4. [**?**, corollary 1 of theorem 1] *Let $f \in \mathbb{Z}_p[X_1, X_2, \ldots, X_m], x \in \mathbb{Z}_p^m$, $n, k \in \mathbb{Z}$. Assume that there exists a natural number j such that $1 \leq j \leq m$,*

$$\frac{\partial f}{\partial X_j}(x) \not\equiv 0 \pmod{p}.$$

Then there exists $y \in \mathbb{Z}_p^m$ such that

$$f(y) = 0 \tag{1}$$

$$y \equiv x \pmod{p} \tag{2}$$

See [**?**] for details.