

$\mathbb{Z}_p, \mathbb{Q}_p$, AND THE RING OF WITT VECTORS

No.05: \mathbb{Z}_p as a local ring.

In this lecture, rings are assumed to be unital, associative and commutative unless otherwise specified.

DEFINITION 5.1. A (unital commutative) ring A is said to be a **local ring** if it has only one maximal ideal.

LEMMA 5.2. *Let A be a ring. Then the following conditions are equivalent:*

- (1) A is a local ring.
- (2) $A \setminus A^\times$ forms an ideal of A .

PROPOSITION 5.3. \mathbb{Z}_p is a local ring. Its maximal ideal is equal to $p\mathbb{Z}_p$.

We may do some “analysis” such as Newton’s method to obtain some solution to algebraic equations.

Newton’s method for approximating a solution of algebraic equation.
Let us solve an equation

$$x^2 = 2$$

in \mathbb{Z}_7 . We first note that

$$3^2 \equiv 2 \pmod{7} \quad (7)$$

hold. So let us put $x_0 = 3 = [0.3]_7$ as the first approximation of the solution. The Newton method tells us that for an approximation x of the equation $x^2 = 2$, a number x' calculated as

$$x' = \frac{1}{2}\left(x + \frac{2}{x}\right)$$

gives a better approximation.

$$x'_0 = \frac{1}{2}([0.3]_7 + [0.3\dot{2}]_7) = [0.3\dot{1}]_7$$

So $[0.3\dot{1}]_7$ is a better approximation of the solution. In order to make the calculation easier, let us choose $x_1 = [0.31]_7$ (instead of x'_0) as a second approximation.

$$x'_1 = \frac{1}{2}([0.31]_7 + 2/[0.31]_7) = \frac{1}{2}([0.31]_7 + [0.3145\dot{2}]_7) \equiv [0.312]_7$$

We choose $x_2 = [0.312]_7$ as a second approximation.

$$x'_2 = \frac{1}{2}([0.312]_7 + 2/[0.3125340662]_7) \doteq [0.31261]_7$$

We choose $x_3 = [0.31261]_7$ as a third approximation.

$$x'_3 \doteq \frac{1}{2}([0.31261]_7 + [0.3126142465066 \dots]_7) \doteq [0.312612124\dots]_7$$

We choose $x_4 = [0.312612124]_7$ as a third approximation.

$$\begin{aligned} x'_4 &= \frac{1}{2}([0.312612124]_7 + [0.312612124565220422662213135351 \dots]_7) \\ &\doteq [0.3126121246621102]_7 \end{aligned}$$

EXERCISE 5.1. Compute $[0.5]_7/[0.11]_7$

EXERCISE 5.2. Find a solution to

$$x^3 \equiv 5 \pmod{11^5}$$

such that $x \equiv 3 \pmod{11}$.

$$\boxed{\mathbb{Q}_p}$$

DEFINITION 5.4. We denote by \mathbb{Q}_p the quotient field of \mathbb{Z}_p .

LEMMA 5.5. *Every non zero element $x \in \mathbb{Q}_p$ is uniquely expressed as*

$$x = p^k u \quad (k \in \mathbb{Z}, u \in \mathbb{Q}_p^\times).$$

We have so far constructed a ring \mathbb{Z}_p and a field \mathbb{Q}_p for each prime p .

PROPOSITION 5.6. *Let p be a prime. Then:*

- (1) \mathbb{Z}_p is a local ring with the unique maximal ideal $p\mathbb{Z}_p$.
- (2)

$$\mathbb{Z}_p/p\mathbb{Z}_p \cong \mathbb{F}_p (= \mathbb{Z}/p\mathbb{Z}).$$

- (3) \mathbb{Z}_p is an integral domain whose quotient field \mathbb{Q}_p is a field of characteristic zero.

With \mathbb{Q}_p and/or \mathbb{Z}_p , we may do some “calculus” such as:

THEOREM 5.7. [?, corollary 1 of theorem 1] *Let $f \in \mathbb{Z}_p[X_1, X_2, \dots, X_m], x \in \mathbb{Z}_p^m, n, k \in \mathbb{Z}$. Assume that there exists a natural number j such that $1 \leq j \leq m$,*

$$\frac{\partial f}{\partial X_j}(x) \not\equiv 0 \pmod{p}.$$

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Then there exists $y \in \mathbb{Z}_p^m$ such that

(1) $f(y) = 0$

(2) $y \equiv x \pmod{p}$

See [?] for details.