ZETA FUNCTIONS. NO.11

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12. Zeta function of graphs.

12.1. **Zeta functions of directed graphs.**

Definition 12.1. A **directed graph** (digraph) is a pair $X^o = (V^o, E^o)$ of:

- a set V^o, whose elements are called vertices or nodes,
- a set E^o called directed edges.
- two maps called "source", "target" from E^o to V^o.

Let $X^o = (V^o, E^o)$ be a directed graph. For each positive integer m, we let N_m to be the number of admissible closed paths in X^o. Then we put

$$Z^o_{X^o}(T) = \exp\left(\sum \frac{1}{m} N_m T^m\right)$$

We define Perron Frobeinus operator $L_{X^o} : C(V^o) \to C(V^o)$ to be

$$L_{X^o}(f)(x) = \sum_{\substack{e \in E^o \\ \mathrm{source}(e)=x}} f(\mathrm{target}(e)).$$

Proposition 12.2.

$$Z^o_{X^o}(T) = \frac{1}{\det(1 - T \cdot L_{X^o})}$$

Exercise 12.1. Let (M, φ) be a dinamical system of a finite set M. Is it possible to define a directed graph $X = (M, E)$ such that its zeta function Z_X coincides with the zeta function of the dinamical system (M, φ)? (Compare the Perron Frobenius matrix L_X with 'the pull back matrix' P_φ.)

12.2. **Ihara zeta function.**

Definition 12.3. A **graph** (undiracted graph) is a pair $X = (V, E)$ of:

- a set V, whose elements are called vertices or nodes,
- a set E called directed edges.
- a map called "(s,t)", from E to $V \times V/\mathfrak{S}_2$.

Let $X=(V,E)$ be a graph. The Ihara zeta function of X is defined by

$$Z(u)=\prod_{\mathfrak{p}\in P}(1-u^{\mathrm{length}(\mathfrak{p})})^{-1},$$

where P denotes the set of prime cycles in X

We define the adjacency operator A as

$$A(f)(x)=\sum_{\substack{e\in E\\ \mathrm{source}(e)=x}} f(\mathrm{target}(e))=\sum_{y\in V}\#\left\{\begin{matrix} e\in E;\\ (\mathrm{s},\mathrm{t})(e)=(x,y)\end{matrix}\right\} f(y).$$

We also define the 'degree operator' D as:

$$D(f)(x)=\deg(x)f(x)$$

where the degree $\deg(x)$ of $x\in V$ is defined as

$$\deg(x)=\#\{e\in E;\mathrm{source}(e)=x\}.$$

THEOREM 12.4.

$$Z(u)=(1-u^2)^{\chi(X)}\det(I-uA+u^2(D-I))^{-1}$$

where $\chi(X)$ is the euler number of X.

12.3. **directed Line graph associated to a graph.** Let $X=(V,E)$ be a graph. Then we define a directed graph $(X_L=(V_L,E_L)$ called **line graph** of X as follows:

(1) $V_L=E$
(2) $E_L^o=\{(e_1,e_2)\in E\times E;\mathrm{target}(e_1)=\mathrm{source}(e_2).,\bar{e}_1\neq e_2\}$.

LEMMA 12.5.

$$\det_{C(E)}(1-u\cdot L_{X_L})=(1-u^2)^{-\chi(X)}\det_{C(V)}(I-uA+u^2(D-I))$$

Reference:

Motoko Kotani and Toshikazu Sunada, Zeta functions of finite graphs, J.Math.Sci.Univ.Tokyo 7(2000) 7-25