COMMUTATIVE ALGEBRA

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06. Length, Hilbert function, Samuel function

LEMMA 6.1 (Artin-Rees). *Let I be an ideal of a Noetherian ring A. Let M be an A module with a submodule N. Then there exists an integer $c > 0$ such that*

$$I^n M \cap N = I^{n-c}(I^c M \cap N)$$

holds for all $n > c$.

THEOREM 6.2 (Krull). *Let A be a ring with an ideal I. Let M be a finite A-module. We set $N = \cap_{k=1}^{\infty} I^k M$. Then there exists $a \in A$ such that $a \equiv 1 \mod I$ and $aN = 0$.*

THEOREM 6.3 (the Krull intersection theorem). *Let A be a Noetherian ring.*

(1) *If I is in the Jacobson radical* $\operatorname{rad} A$ *of A, then for any finite A-module M, we have $\cap_n I^n M = 0$. Furthermore, for any submodule N of M, we have $\cap_n I^n M \cap N = 0$.*
(2) *If A is an integral domain and I is a proper ideal of A, then we have $\cap_n I^n = 0$.*

PROPOSITION 6.4. *Let A be a local ring. The following conditions are equivalent:*

(1) $l(A) < \infty$ *(which is also equivalent to saying that $d(A) = 0$ or that $\delta(A) = 0$).*
(2) $\dim(A) = 0$.
(3) *Any descending chain*

$$I_0 \supset I_1 \supset I_2 \supset \dots$$

of ideals of A stops.

LEMMA 6.5. *Let A be a ring with an ideal I. Let M be an A/I-module. then we may (of course) consider M as an A-module. The dimensions* $\dim(M), d(M), \delta(M)$ *are irrelevant of whether we consider M as an A/I-module or as an A-module.*

LEMMA 6.6. *Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of finite A-modules over a Noetherian local ring A. Then:*

(1) $d(M) = \max(d(M'), d(M''))$.
(2) *For any ideal I of definition of A, The leading coefficient of $\chi^I_M - \chi^I_{M''}$ coincides with that of $\chi^I_{M'}$.*

THEOREM 6.7. *Let $(A, \mathfrak{m})$ be a d-dimensional regular local ring with the residue field $k = A/\mathfrak{m}$. Then*

$$\operatorname{gr}_{\mathfrak{m}}(A) \cong k[X_1, \dots, X_d].$$