CATEGORIES, ABELIAN CATEGORIES AND COHOMOLOGIES.

YOSHIFUMI TSUCHIMOTO

Adjoining inverses

DEFINITION 10.1. Let A be a commutative ring. Let S be its subset. We say that S is multiplicative if

(1) $1 \in S$
(2) $x, y \in S \implies xy \in S$

holds.

DEFINITION 10.2. Let S be a multiplicative subset of a commutative ring A. Then we define $A[S^{-1}]$ as

$$A[\{X_s; s \in S\}]/(\{sX_s - 1; s \in S\})$$

where in the above notation X_s is a indeterminate prepared for each element $s \in S$.) We denote by ι_S a canonical map $A \to A[S^{-1}]$.

LEMMA 10.3. *Let S be a multiplicative subset of a commutative ring A. Then the ring $B = A[S^{-1}]$ is characterized by the following property:*

Let C be a ring, $\varphi : A \to C$ be a ring homomorphism such that $\varphi(s)$ is invertible in C for any $s \in S$. Then there exists a unique ring homomorphism $\psi = \phi[S^{-1}] : B \to C$ such that

$$\varphi = \psi \circ \iota_S$$

holds.

COROLLARY 10.4. *Let S be a multiplicative subset of a commutative ring A. Let I be an ideal of A given by*

$$I = \{x \in I; \exists s \in S \text{ such that } sx = 0\}$$

Then (1) I is an ideal of A. Let us put $\bar{A} = A/I$, $\pi : A \to \bar{A}$ the canonical projection. Then:
(2) $\bar{S} = \pi(S)$ is multiplicatively closed.
(3) We have

$$A[S^{-1}] \cong \bar{A}[\bar{S}^{-1}]$$

(4)$\iota_{\bar{S}} : \bar{A} \to \bar{A}[\bar{S}^{-1}]$ is injective.

DEFINITION 10.5. Let S be a multiplicative subset of a commutative ring A. Let M be an A-module we may define $S^{-1}M$ as

$$\{(m/s); m \in M, s \in S\}/\sim$$

where the equivalence relation $\sim$ is defined by

$$(m_1/s_1) \sim (m_2/s_2) \iff t(m_1 s_2 - m_2 s_1) = 0 \quad (\exists t \in S).$$

We may introduce a $S^{-1}A$-module structure on $S^{-1}M$ in an obvious manner.

$S^{-1}M$ thus constructed satisfies an universality condition which the reader may easily guess.

LEMMA 10.6. *Let A be a commutative ring. Let M be an A-module. Then we have a canonical isomorphism of A_S module*

$$A_S \otimes_A M \cong M_S.$$

We may also localize categories, but we need to deal with non commutativity of composition. To simplify the situation we only deal with a localization with some nice properties as follows:

(1) (a) $s, t \in \Sigma \implies st \in \Sigma$
    (b) $X \in \mathrm{Ob}(\mathcal{C}) \implies 1_X \in \Sigma$.

(2) Let $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$. Let $u \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, $s \in \mathrm{Hom}_{\mathcal{C}}(Z, Y) \cap \Sigma$. Then there exist $W \in \mathrm{Ob}(\mathcal{C})$ and morphisms $v \in \mathrm{Hom}_{\mathcal{C}}(W, Z)$, and $t \in \mathrm{Hom}_{\mathcal{C}}(W, X) \cap \Sigma$ such that the diagram

$$\begin{array}{ccc} W & \xrightarrow{v} & Z \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle s} \\ X & \xrightarrow{u} & Y \end{array}$$

commutes.

In a simpler (but not rigorous) words, for each "composable $s^{-1}u$", there exists v, t such $s^{-1}u = vt^{-1}$. Similarly, for each composable us^{-1}, there exists v, t such that $us^{-1} = t^{-1}v$ holds.

(3) Let $X, Y \in \mathrm{Ob}(\mathcal{C})$, $u, v \in Hom_{\mathcal{C}}(X, Y)$. Then the following conditions are equivalent:
    (a) There exists $Y' \in \mathrm{Ob}(\mathcal{C})$ and $s \in \mathrm{Hom}_{\mathcal{C}}(Y, Y') \cap \Sigma$ such that $su = sv$.
    (b) There exists $X' \in \mathrm{Ob}(\mathcal{C})$ and $t \in \mathrm{Hom}_{\mathcal{C}}(Y, Y') \cap \Sigma$ such that $ut = vt$.

(4) If $s \in \Sigma$ and if $su \in \Sigma$ then $u \in \Sigma$.

LEMMA 10.7. *Let Σ be a family of morphisms in $\mathcal{C}$ which satisfies the properties above. Then one may construct a localization of $\mathcal{C}_\Sigma$ with respect to Σ. Furthermore, if $\mathcal{C}$ is additive, then $\mathcal{C}_\Sigma$ is also additive.*