CATEGORIES, ABELIAN CATEGORIES AND COHOMOLOGIES.

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Resolutions and derived functors

We recommend the book of Lang [1] as a good reference. The treatment here follows the book for the most part.

THEOREM 7.1. *Let $\mathcal{C}_1$ be an abelian category with enough injectives, and let $F : \mathcal{C}_1 \to \mathcal{C}_2$ be a covariant additive left functor to another abelian category $\mathcal{C}_2$. Then:*

(1) $F \cong R^0F$.

(2) *For each short exact sequence*

$$0 \to M' \to M \to M'' \to 0$$

and for each $n \geq 0$ there is a natural homomorphism

$$\delta^n : R^nF(M'') \to R^{n+1}F(M)$$

such that we obtain a long exact sequence

$$\cdots \to R^nF(M') \to R^nF(M) \to R^nF(M'') \xrightarrow{\delta^n} R^{n+1}F(M') \to \dots.$$

(3) *δ is natural. That means, for a morphism of short exact sequences*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & N' & \longrightarrow & N & \longrightarrow & N'' & \longrightarrow & 0 \end{array}$$

the δ's give a commutative diagram:

$$\begin{array}{ccc} R^nF(M'') & \xrightarrow{\delta^n} & R^{n+1}F(M') \\ \downarrow & & \downarrow \\ R^nF(N'') & \xrightarrow{\delta^n} & R^{n+1}F(N') \end{array}$$

(4) *For each injective objective object I of A and for each $n > 0$ we have $R^nF(I)$.*

LEMMA 7.2. *For any exact sequence $0 \to M' \to M \to M'' \to 0$ of objects in $\mathcal{C}_1$, There exists injective resolutions $I_{M'}, I_M, I_{M''}$ of M', M, M'' respectively and a commutative diagram*

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I_{M'} & \longrightarrow & I_M & \longrightarrow & I_{M''} & \longrightarrow & 0 \end{array}$$

such that the diagram of resolutions is exact.

DEFINITION 7.3. Let F be a left exact additive functor. An object X is called F-acyclic if $R^nF(X) = 0$ for all $n > 0$.

Theorem 7.4. *Let*

$$0 \to M \to X^0 \to X^1 \to X^2 \to \dots$$

be a resolution of M by F-acyclics. Let

$$0 \to M \to I^0 \to I^1 \to I^2 \to \dots$$

be an injective resolution. Then there exists a morphism of complexes $X \to I$ extending the identity on M, and this morphism induces an isomorphism

$$H^n F(X) \cong H^n(F(I)) = R^n F(M)$$

for all $n \geq 0$.

Note: Our notation of denoting complexes such as I_M differs from that in [1].

The book of Grivel [2] is also a good reference for our future arguments.

References

[1] S. Lang, *Algebra (graduate texts in mathematics)*, Springer Verlag, 2002.

[2] P.P.Grivel, *Catégorie dérivées et foncteurs dérivés*, In: Algebraic D-modules, Perspectives in mathematics **2** (1997), 1–108.