CATEGORIES, ABELIAN CATEGORIES AND COHOMOLOGIES.

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Categories of modules. We first review definition and some basic properties of modules.

DEFINITION 4.1. Let R be a (unital associative) ring. A set M is said to be an **R-module** if there is given a binary map ("action")

$$R \times M \ni (r, m) \mapsto r.m \in M$$

such that the following properties hold.

(1) $\forall r \in R \forall m_1, \forall m_2 \in M \quad r.(m_1 + m_2) = r.m_1 + r.m_2$.
(2) $\forall r_1, \forall r_2 \in R \forall m \in M \quad (r_1 + r_2).m = r_1.m + r_2.m$.
(3) $\forall m \in M \quad 1_R.m = m$.
(4) $\forall r_1, \forall r_2 \in R \forall m \in M \quad (r_1 r_2).m = r_1.(r_2.m)$

DEFINITION 4.2. Let R be a (unital associative) ring. A map φ from an R-module M to another R-module N is an **R-module homomorphism** if the following conditions are satisfied.

(1) φ is additive. That means, we have
$$\forall m_1 \forall m_2 \in M \quad \varphi(m_1 + m_2) = \varphi(m_1) + \varphi(m_2).$$
(2) φ preserves the R-action. That means,
$$\forall r \in R \forall m \in M \quad \varphi(r.m) = r.\varphi(m).$$

PROPOSITION 4.3. *For any given ring R, The category (R-mod) of R-modules is an abelian category.*

EXERCISE 4.1. Let A be a 2×2 matrix

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

over $\mathbb{C}$. We define a structure of $\mathbb{C}[X]$-module on $V = \mathbb{C}^2$ by putting

$$f(X).v = f(A)v \qquad (v \in \mathbb{C}^2)$$

Then:

(1) Show that V has a proper $\mathbb{C}[X]$-submodule W. (That means, $\mathbb{C}[X]$ submodule such that $W \neq V, 0$.
(2) Show that there is no other proper submodule of V.

DEFINITION 4.4. A **cochain complex** in an abelian category $\mathcal{C}$ is a sequence of objects and morphisms in $\mathcal{C}$

$$C^\bullet : \dots \overset{d^{n-1}}{\to} C^n \overset{d^n}{\to} C^{n+1} \overset{d^{n+1}}{\to} \dots$$

such that $d^n \circ d^{n-1} = 0$.

Cohomology objects of the cochain complex are

$$H^n(C^\bullet) = \mathrm{Ker}(d^n) / \mathrm{Image}(d^{n-1}).$$