$\mathbb{Z}_p$, $\mathbb{Q}_p$, AND THE RING OF WITT VECTORS

No.09: Witt algebras (3) The ring of p-adic Witt vectors

Definition 9.1. Let A be any commutative ring. Let n be a positive integer. Let us define additive operators V_n, F_n on $\mathcal{W}_1(A)$ by the following formula.

$$V_n(f(T)) = f(T^n).$$

$$F_n(f(T)) = \prod_{\zeta \in \mu_n} f(\zeta T^{1/n})$$

(The latter definition is a formal one. That means, F_n is actually defined to be the unique continuous additive map which satisfies

$$F_n(1 - aT^l) = (1 - a^{m/l} T^{m/n})^{ln/m} \qquad (m = l.c.m(n, l)).$$

)

Lemma 9.2. *Let p be a prime number. Let A be acommutative ring of characteristic p. Then:*

(1) *We have*

$$F_p(f(T)) = (f(T^{1/p}))^p \qquad (\forall f \in \mathcal{W}_1(A)).$$

in partucular, F_p is an algebra endomorphism of $\mathcal{W}_1(A)$ in this case.

(2)

$$V_p(F_p(f) = F_p(V_p(f)) = (f(T))^p = p \boxdot f(T) (= \overbrace{f(T) \boxplus \cdots \boxplus f(T)}^{p}).$$

Definition 9.3. Let A be any commutative ring. Let p be a prime number. We denote by

$$\mathcal{W}^{(p)}(A) = A^{\mathbb{N}}.$$

and define

$$\pi_p : \mathcal{W}_1(A) \to \mathcal{W}^{(p)}(A)$$

by

$$\pi_p \left(\sum_{j=1}^{\infty}{}^{\boxplus} (1 - x_j T^j) \right) = (x_1, x_p, x_{p^2}, x_{p^3} \ldots).$$

Lemma 9.4. *Let us define polynomials $\alpha_j(X, Y) \in \mathbb{Z}[X, Y]$ as follows.*

$$(1 - xT)(1 - yT) = \prod_{j=1}^{\infty} (1 - \alpha_j(x, y) T^j).$$

Then we have the following rule for "carry operation":

$$(1 - xT^n) \boxplus (1 - yT^n) = \sum_{j=1}^{\infty}{}^{\boxplus} (1 - \alpha_j(x, y) T^{jn}).$$

PROPOSITION 9.5. *There exist unique binary operators* $\boxplus$ *and* $\boxtimes$ *on* $\mathcal{W}^{(p)}(A)$ *such that the following diagrams commute.*

$$\begin{array}{ccc} \mathcal{W}_1(A) \times \mathcal{W}_1(A) & \xrightarrow{\boxplus} & \mathcal{W}_1(A) \\ {\scriptstyle \pi_p}\downarrow & & {\scriptstyle \pi_p}\downarrow \\ \mathcal{W}^{(p)}(A) \times \mathcal{W}^{(p)}(A) & \xrightarrow{\boxplus} & \mathcal{W}^{(p)}(A) \end{array}$$

$$\begin{array}{ccc} \mathcal{W}_1(A) \times \mathcal{W}_1(A) & \xrightarrow{\boxtimes} & \mathcal{W}_1(A) \\ {\scriptstyle \pi_p}\downarrow & & {\scriptstyle \pi_p}\downarrow \\ \mathcal{W}^{(p)}(A) \times \mathcal{W}^{(p)}(A) & \xrightarrow{\boxtimes} & \mathcal{W}^{(p)}(A) \end{array}$$

PROOF. Using the rule as in the previous lemma, we see that addition descends to an addition of $\mathcal{W}^{(p)}(A)$. It is easier to see that the multiplication also descends.

□

DEFINITION 9.6. For any commutative ring A, elements of $W^{(p)}(A)$ are called p**-adic Witt vectors** over A. The ring $(W^{(p)}(A), \boxplus, \boxtimes)$ is called **the ring of** p**-adic Witt vectors** over A.