TOPICS IN NON COMMUTATIVE ALGEBRAIC GEOMETRY AND CONGRUENT ZETA FUNCTIONS (PART II). REVIEW OF COMMUTATIVE (USUAL) AFFINE SCHEMES.

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...I used to eat there. Really good noodles.

Neo, ca.1999

1. (Usual) affine schemes

In Part I of this talk, we have learned that Weyl algebras have a lot of finite dimensional representations if the characteristic of the base field is non zero. We would like to formulate this in terms of schemes.

Before we do this, let us review definitions and preliminaries of usual (that means, commutative) affine schemes. We do this in our way. That means, by focusing on representations, especially on irreducible ones.

1.1. **Spectrums of a commutative ring.** Let A be a commutative ring. (Recall that we always assume ring to be unital associative.) Amazingly enough(!?), any element z in A is central. As we have seen in the Schur's Lemma, for any "finite dimensional" irreducible representation ρ of A, $\rho(z)$ should be a scalar. Thus we see that any irreducible "finite dimensional" irreducible representation of A should be one dimensional. Though this argument does not make sense when A has no restriction such as "A is finitely generated over a field", we may begin by considering a one-dimensional representation of A. That means, a ring homomorphism

$$\rho : A \to K$$

where K is a field. One knows that

(1) $A/\operatorname{Ker}(\rho)$ is an integral domain. That means, it has no zero-divisor other than zero. (In this sense, $\operatorname{Ker}(\rho)$ is said to be a prime ideal of A.)
(2) ρ is decomposed in the following way.

$$A \to A/\operatorname{Ker}(\rho) \to Q(A/\operatorname{Ker}(\rho)) \to K$$

where $Q(B)$ is the field of fractions of a ring B.

With a suitable definition of "equivalence" of such representations, we may identify equivalence class of representation with the kernel $\mathrm{Ker}(\rho)$.

In other words, we are interested in prime ideals.

DEFINITION 1.1. Let A be a commutative ring. Then we define the set $\mathrm{Spec}(A)$ of spectrum of A as the set of prime ideals of A.

We note that for any $\mathfrak{p} \in \mathrm{Spec}(A)$, we have a ring homomorphism ("representation associated to $\mathfrak{p}$")$\rho_\mathfrak{p}$ defined by

$$\rho_\mathfrak{p} : A \to A/\mathfrak{p} \to Q(A/\mathfrak{p}).$$

Since $A/\mathfrak{p} \to Q(A/\mathfrak{p})$ is an inclusion, we may say, by abuse of language, that the value of an element $f \in A$ under the representation $\rho_\mathfrak{p}$ is equal to $f \pmod{\mathfrak{p}}$. We note further that

$$\mathfrak{p} = \{f \in A; \rho_\mathfrak{p}(f) = 0\}$$

holds.

Let us now define a topology on $\mathrm{Spec}(A)$.

DEFINITION 1.2. Let A be a commutative ring. For any $f \in A$, we define a subset O_f of $\mathrm{Spec}(A)$ defined by

$$O_f = \{\mathfrak{p} \in \mathrm{Spec}(A); \rho_\mathfrak{p}(f) \neq 0\}.$$

LEMMA 1.3. *Let A be a commutative ring. Then we have*

$$O_f \cap O_g = O_{fg}$$

for any $f, g \in A$. $\{O_f; f \in A\}$. Thus we may introduce a topology on $\mathrm{Spec}(A)$ *whose open sets are unions of various O_f.*

PROOF.

$$O_f \cap O_g = \{\mathfrak{p}; \rho_\mathfrak{p}(f) \neq 0 \text{ and } \rho_\mathfrak{p}(g) \neq 0\} = \{\mathfrak{p}; \rho_\mathfrak{p}(fg) \neq 0\}$$

□

DEFINITION 1.4. The topology defined in the preceding Lemma is called the Zariski topology of $\mathrm{Spec}(A)$.

In Part II, we always equip $\mathrm{Spec}(A)$ with the Zariski topology. Thus for any commutative ring A, we may always associate a topological space $\mathrm{Spec}(A)$.

1.2. **ring homomorphism and spectrum.**

LEMMA 1.5. *Let A,B be two ring homomorphisms. Let*

$$\alpha : A \to B$$

be a ring homomorphism (which we always assume to be unital).
Then we have a associate map

$$\operatorname{Spec}(\alpha) : \operatorname{Spec}(B) \to \operatorname{Spec}(A)$$

defined by

$$\operatorname{Spec}(\alpha)(\mathfrak{p}) = \alpha^{-1}(\mathfrak{p}) \qquad (\forall \mathfrak{p} \in \operatorname{Spec}(B)).$$

The map $\operatorname{Spec}(\alpha)$ *has the following properties.*

(1)
$$\operatorname{Spec}(\alpha)(\mathfrak{p}) = \{f \in A; \rho_{\mathfrak{p}}(\alpha(f)) = 0\}$$

(2)
$$\operatorname{Spec}(\alpha)^{-1}(O_f) = O_{\alpha(f)}$$

for any $f \in A$.

(3) $\operatorname{Spec}(\alpha)$ *is continuous.*

1.3. **localization of a commutative ring.** .

DEFINITION 1.6. Let f be an element of a commutative ring A. Then we define the localization A_f of A with respect to f as a ring defined by

$$A_f = A[X]/(Xf - 1)$$

where X is a indeterminate.

In the ring A_f, the residue class of X plays the role of the inverse of f. Therefore, we may write $A[1/f]$ instead of A_f if there is no confusion.

One may define localization in much more general situation. The reader is advised to read standard books on commutative algebras.

LEMMA 1.7. *Let f be an element of a commutative ring A. Then there is a canonically defined homeomorphism between O_f and* $\operatorname{Spec}(A_f)$. *(It is usual to identify these two via this homeomorphism.)*

PROOF. Let $i_f : A \to A_f$ be the natural homomorphism. We have already seen that we have a continuous map

$$\operatorname{Spec}(i_f) : \operatorname{Spec}(A_f) \to \operatorname{Spec}(A).$$

We need to show that it is injective, and that it gives a homeomorphism between $\operatorname{Spec}(A_f)$ and O_f.

Let us do this by considering representations.

(1) $\mathfrak{p} \in \operatorname{Spec}(A)$ corresponds to a representation $\rho_{\mathfrak{p}}$.

(2) $\mathfrak{q} \in \operatorname{Spec}(A_f)$ corresponds to a representation $\rho_{\mathfrak{q}}$.
(3) $\operatorname{Spec}(i_f)$ corresponds to a restriction map $\rho \mapsto \rho \circ i_f$.

Now, for any $\mathfrak{p} \in \operatorname{Spec}(A)$, $\rho_{\mathfrak{p}}$ extends to A_f if and only if the image $\rho_{\mathfrak{p}}(f)$ of f is invertible, that means, $\rho_{\mathfrak{p}}(f) \neq 0$. In such a case, the extension is unique. (We recall the fact that the inverse of an element of a field is unique.)

It is easy to prove that $\operatorname{Spec}(i_f)$ is a homeomorphism. □

Let A be a ring. Let $f \in A$. It is important to note that each element of A_f is written as a "fraction"

$$\frac{x}{f^k} \qquad (x \in A; k \in \mathbb{N}).$$

One may introduce A_f as a set of such formal fractions which satisfy ordinary computation rules. In precise, we have the following Lemma.

LEMMA 1.8. *Let A be a ring, f be its element. Let us consider the following set*

$$S = \{(x, f^k); x \in A; k \in \mathbb{N}\}.$$

We introduce on S the following equivalence law.

$$(x, f^k) \sim (y, f^l) \iff (yf^k - xf^l)f^N = 0 \qquad (\exists N \in \mathbb{N})$$

Then we may obtain a ring structure on $S/\sim$ by introducing the following sum and product.

$$(x/f^k) + (y/f^l) = (xf^l + yf^k/f^{k+l})$$
$$(x/f^k)(y/f^l) = (xy/f^{k+l})$$

where we have denoted by (x/f^k) the equivalence class of $(x, f^k) \in S$.

COROLLARY 1.9. *Let A be a ring, f be its element. Then we have $A_f = 0$ if and only if f is nilpotent.*

Likewise, for any A-module M, we may define M_f as a set of formal fractions

$$\frac{m}{f^k} \qquad (m \in M; k \in \mathbb{N}).$$

which satisfy certain computation rules.

1.3.1. *Existence of a point.*

LEMMA 1.10. *Let A be a ring. If $A \neq 0$ (which is equivalent to saying that $1_A \neq 0_A$), then we have* $\operatorname{Spec}(A) \neq \emptyset$.

PROOF. Assume $A \neq 0$. Then by Zorn's lemma we always have a maximal ideal $\mathfrak{m}$ of A. A maximal ideal is a prime ideal of A and is therefore an element of $\operatorname{Spec}(A)$. □

LEMMA 1.11. *Let A be a ring, f be its element. We have $O_f = \emptyset$ if and only if f is nilpotent.*

PROOF. We have already seen that $A_f = 0$ if and only if f is nilpotent. (Corollary 1.9). Since O_f is homeomorphic to $\mathrm{Spec}(A_f)$, we have the desired result. □

1.4. **Zariski topology on affine schemes.** In Definition 1.4, we have already defined the Zariski topology on an affine scheme $\mathrm{Spec}(A)$.

In this section we describe some of its properties. Due to a limitation on the time, we shall only give a very short account on this. See [2] for an excellent explanations on how the Zariski topology and properties of rings are related to each other.

1.4.1. *compactness.*

THEOREM 1.12. *For any commutative ring A, the spectrum* $\mathrm{Spec}(A)$ *of A (equipped with the Zariski topology) is a compact set.*

PROOF. Let $\mathfrak{U} = \{U_\lambda\}$ be an open covering of $\mathrm{Spec}(A)$. We want to find a finite subcovering of $\mathfrak{U}$.

For any $x \in \mathrm{Spec}(A)$, we have a index λ_x and an open subset O_{f_x} of U_{λ_x} such that

$$x \in O_{f_x} \subset U_{\lambda_x}$$

holds. Replacing $\mathfrak{U}$ by $\{U_{\lambda_x}\}_{x \in \mathrm{Spec}(A)}$ if necessary, we may assume each U_λ is of the form O_{f_λ} for some $f_\lambda \in A$.

Now,

$$\cup O_{f_\lambda} = \mathrm{Spec}(A)$$

implies that

$$\forall x \in \mathrm{Spec}(A) \exists \lambda \text{ such that } \rho_x(f_\lambda) \neq 0 \quad (\text{that means, } f_\lambda \notin x.)$$

Now we would like to show from this fact that the ideal I defined by

$$I = \{f_\lambda\}_{\lambda \in \Lambda}$$

is equal to A. Assume the contrary. Using Zorn's lemma we may always obtain an maximal ideal $\mathfrak{m}$ of A which contains I. This is a contradiction to the fact mentioned above.

Thus we have proved that $I = A$. In particular, we may find a relation

$$1 = \sum_{j=0}^{N} a_j f_{\lambda_j}$$

for some positive integer N, index sets $\{\lambda_j\}_{j=0}^N$, and elements $a_j \in A$. This clearly means that

$$\bigcup_{j=0}^{N} O_{f_{\lambda_j}} = \operatorname{Spec}(A)$$

as required. □

1.4.2. *closed subsets.*

DEFINITION 1.13. Let A be a commutative ring. Let S be a subset of A. Then we define $V(S)$ as follows.

$$V(S) = \bigcap_{f \in S} \complement(O_f)$$

It is a closed subset of Spec(A).

LEMMA 1.14. *Let A be a commutative ring. Then for any subset S of A, we have the following.*

(1)
$$V(S) = \{x \in \operatorname{Spec}(A); \rho_x(f) = 0 \forall f \in S\}$$
(2) $V(S) = V(I)$ *where* $I = A.S$ *is the ideal of* A *generated by* S.
(3)
$$V(I) = \{x \in \operatorname{Spec}(A); I \subset x\}$$

PROOF. clear from the definition. □

Thus a closed set in Spec(A) is of the form $V(I)$ for some ideal I.

LEMMA 1.15. *For any ideals I, J of a commutative ring A, we have the following.*

(1) $V(I + J) = V(I) \cap V(J)$.
(2) $V(IJ) = V(I) \cup V(J)$.
(3) $V(I) = \emptyset \iff I = A$.
(4) $V(I) = \operatorname{Spec}(A) \iff$ *any element of I is nilpotent.*

PROOF. (3): if $I \subsetneq A$, then by the Zorn's lemma we obtain a maximal ideal $\mathfrak{m}$ which contains I. Since maximal ideals are prime, we have

$$V(I) \ni \mathfrak{m}.$$

Thus $V(I)$ is not empty. The converse is obvious.
(4)is a consequence of Lemma 1.11. □

The reader may easily see that the compactness of Spec(A) (Theorem 1.12) is proved in a more easier way if we have used the terms of closed sets and "finite intersection property".

The author cannot help but mentioning little more how the topology of Spec(A) and the structure of A related to each other.

Though the following statements may never be used in this talk (at least in the near future), we would like to record the statement and its proof.

THEOREM 1.16. *Let A be a ring.*

(1) *Assume* Spec(A) *is not connected so that it is divided into two disjoint closed sets $V(I)$ and $V(J)$.*

$$\operatorname{Spec}(A) = V(I) \cup V(J), \quad V(I) \cap V(J) = \emptyset.$$

Then we have elements $p_1, p_2 \in A$ such that

(a) $p_1^2 = p_1, p_2^2 = p_2, p_1 + p_2 = 1$

(b) *A is a product of algebras Ap_1, Ap_2*

(c) *$\rho_x(p_1) = 1$ for all $x \in V(J)$.*

(d) *$\rho_x(p_1) = 0$ for all $x \in V(I)$.*

(2) *Conversely, if the ring A has elements p_1, p_2 which satisfy (a)-(b) above, then* Spec(A) *is divided into two disjoint closed sets.*

PROOF. (1) Since $V(I)$ and $V(J)$ is disjoint, we have

$$V(I + J) = V(I) \cap V(J) = \emptyset$$

Thus $I + J = A$. It follows that there exists $a_1 \in I$ and $a_2 \in J$ such that $a_1 + a_2 = 1$. On the other hand,

$$V(IJ) = V(I) \cup V(J) = \operatorname{Spec}(A)$$

implies that any element of IJ is nilpotent. Let N be a positive integer such that

$$(a_1 a_2)^N = 0$$

holds. Then by expanding the equation

$$(a_1 + a_2)^{2N} = 1,$$

we obtain an equation of the following form

$$u_1 a_1^N + u_2 a_2^N = 1 \qquad (\exists u_1, u_2 \in A)$$

Indeed, we have

$$\sum_{j=N+1}^{2N} \left(\binom{2N}{j} a_1^{j-N} a_2^{2N-j} \right) a_1^N + \sum_{j=0}^{N-1} \left(\binom{2N}{j} a_1^j a_2^{N-j} \right) a_2^N = 1.$$

Now let us put $p_1 = u_1 a_1^N, p_2 = u_2 a_2^N$. They satisfy

$$p_1 + p_2 = 1, \quad p_1 p_2 = 0, \quad p_1 \in I, \quad p_2 \in J.$$

Then it is easy to verify that the elements p_1, p_2 satisfy the required properties. The converse is easier and is left to the reader. □

1.5. **sheaves.** Affine spectrum $\operatorname{Spec}(A)$ of a ring A carries one more important structure. Namely, its structure sheaf.

We will firstly review some definitions and first properties of sheaves.

To illustrate the idea, we recall an easy lemma in topology.

LEMMA 1.17 (Gluing lemma). *Let X, Y be a topological spaces. Let $\{U_\lambda\}_{\lambda \in \Lambda}$ be an open covering of X.*

(1) *If we are given a collection of continuous maps $\{f_\lambda : U_\lambda \to Y\}_{\lambda \in \Lambda}$ such that*

$$f_\lambda|_{U_\lambda \cap U_\mu} = f_\mu|_{U_\lambda \cap U_\mu}$$

holds for any pair $(\lambda, \mu) \in \Lambda^2$, then we have a unique continuous map $f : X \to Y$ such that

$$f|_{U_\lambda} = f_\lambda$$

holds for any $\lambda \in \Lambda$.

(2) *Conversely, if we are given a continuous map $f : X \to Y$, then we obtain a collection of maps $\{f_\lambda : U_\lambda \to Y\}_{\lambda \in \Lambda}$ by restriction.*

PROOF. (1) It is easy to verify that we have a well-defined map

$$f : X \to Y$$

with

$$f|_{U_\lambda} = f_\lambda.$$

The continuity of f is proved by verifying that the inverse image of any open set $V \subset Y$ by f is open in X. □

1.5.1. *A convention.* Before proceeding further, we employ the following convention.

For an open covering $\{U_\lambda\}_{\lambda \in \Lambda}$ of a topological space X, we write

$$U_{\lambda\mu} = U_\lambda \cap U_\mu, \qquad U_{\lambda\mu\nu} = U_\lambda \cap U_\mu \cap U_\nu,$$

and so on.

1.5.2. *presheaves.* We first define presheaves.

DEFINITION 1.18. Let X be a topological space. We say "a presheaf $\mathcal{F}$ of rings over X is given" if we are given the following data.

(1) For each open set $U \subset X$, a ring denoted by $\mathcal{F}(U)$. (which is called the ring of sections of $\mathcal{F}$ on U.)
(2) For each pair U, V of open subsets of X such that $V \subset U$, a ring homomorphism (called restriction)
$$\rho_{VU} : \mathcal{F}(U) \to \mathcal{F}(V).$$

with the properties

(1) $\mathcal{F}(\emptyset) = 0$.
(2) We have $\rho_{U,U} =$ identity for any open subset $U \subset X$.
(3) We have
$$\rho_{WV}\rho_{VU} = \rho_{WV}$$
for any open sets $U, V, W \subset X$ such that $W \subset V \subset U$.

1.5.3. *sheaves.*

DEFINITION 1.19. Let X be a topological space. A presheaf $\mathcal{F}$ of rings over X is called a sheaf if for any open set $U \subset X$ and for any open covering $\{U_\lambda\}_{\lambda \in \Lambda}$ of U, it satisfies the following conditions.

(1) ("Locality") If there is given a local section $f, g \in \mathcal{F}(U)$ such that
$$\rho_{U_\lambda U}(f) = \rho_{U_\lambda U}(g)$$
holds for all $\lambda \in \Lambda$, then we have $f = g$
(2) ("Gluing lemma"). If there is given a collection of sections $\{f_\lambda\}_{\lambda \in \Lambda}$ such that
$$\rho_{U_{\lambda\mu} U_\lambda}(f_\lambda) = \rho_{U_{\lambda\mu} U_\mu}(f_\mu)$$
holds for any pair $(\lambda, \mu) \in \Lambda^2$, then we have a section $f \in \mathcal{F}(U)$ such that
$$\rho_{U_\lambda U}(f) = f_\lambda$$
holds for all $\lambda \in \Lambda$.

We may similarly define sheaf of sets, sheaf of modules, etc.

LEMMA 1.20. *Let X be a topological set with an open base $\mathfrak{U}$. To define a sheaf $\mathcal{F}$ over X we only need to define $\mathcal{F}(U)$ for every member U of $\mathfrak{U}$ and check the sheaf axiom for open bases. In precise, given such data, we may always find a unique sheaf $\mathcal{G}$ on X such that $G(U) \cong F(U)$ holds in a natural way. (That means, the isomorphism commutes with restrictions wherever they are defined.)*

Proof. Let $\mathcal{F}$ be such. For any open set $U \subset X$, we define a presheaf $\mathcal{G}$ by the following formula.

$$\mathcal{G}(U) = \left\{ (s_V) \in \prod_{V \in \mathfrak{U}, V \subset U} \mathcal{F}(V); \begin{array}{c} \rho_{WV}(s_V) = s_W \text{ for any } V, W \in \mathfrak{U} \\ \text{with the property } W \subset V \subset U. \end{array} \right\}$$

Restriction map of $\mathcal{G}$ is defined in an obvious manner.

Then it is easy to see that $\mathcal{G}$ satisfies the sheaf axiom and that

$$\mathcal{G}(U) \cong \mathcal{F}(U)$$

holds for any $U \in \mathfrak{U}$ in a natural way.

□

Lemma 1.21. *Let A be a ring.*

(1) *We have a sheaf $\mathcal{O}$ of rings on* $\operatorname{Spec}(A)$ *which is defined uniquely by the property*

$$\mathcal{O}(O_f) = A_f \qquad (\forall f \in A)$$

(2) *For any A-module M we have a sheaf $\tilde{M}$ of modules on* $\operatorname{Spec}(A)$ *which is defined uniquely by the property*

$$\tilde{M}(O_f) = M_f \qquad (\forall f \in A)$$

(3) *For any A-module M, the sheaf $\tilde{M}$ is a sheaf of $\mathcal{O}$-modules on* $\operatorname{Spec}(A)$*. That means, it is a sheaf of modules over* $\operatorname{Spec}(A)$ *with an additional $\mathcal{O}$-action (which is defined in an obvious way.)*

Proof. We prove (2).

From the previous Lemma, we only need to prove locality and gluing lemma for open sets of the form O_f. That means, in proving the properties (1) and (2) of Definition 1.19, we may assume that $U_\lambda = O_{f_\lambda}, U = O_f$ for some elements $f_\lambda, f \in A$.

Furthermore, in doing so we may use the identification $O_f \approx \operatorname{Spec} A_f$. By replacing A by A_f, this means that we may assume that $O_f = \operatorname{Spec}(A)$.

To sum up, we may assume

$$U = \operatorname{Spec}(A), U_\lambda = O_{f_\lambda}.$$

To simplify the notation, in the rest of the proof, we shall denote by

$$i_\lambda : M \to M_{f_\lambda}$$

the canonical map which we have formerly written i_{f_λ}. Furthermore, for any pair λ, μ of the index set, we shall denote by $i_{\lambda\mu}$ the canonical map

$$i_{\lambda\mu} : M \to M_{f_\lambda f_\mu}.$$

Locality: Compactness of $\operatorname{Spec}(A)$ (Theorem 1.12) implies that there exist finitely many open sets $\{O_{f_j}\}_{j=1}^k$ among U_λ such that $\cup_{j=1}^k O_{f_j} = \operatorname{Spec}(A)$. In particular there exit elements $\{c_j\}_{j=1}^k$ of A such that

$$c_1 f_1 + c_2 f_2 + \cdots + c_k f_k = 1 \tag{PU}$$

holds.

Let $m, n \in M$ be elements such that

$$i_j(m) = i_j(n) \qquad (\text{ in } M_{f_j}.)$$

With the help of the "module version" of Lemma 1.8, we see that for each j, there exist positive integers N_j such that

$$f_j^{N_j}(m-n) = 0$$

holds for all $j \in \{1, 2, 3, \ldots, k\}$. Let us take the maximum N of $\{N_j\}$. It is easy to see that

$$f_j^N(m-n) = 0$$

holds for any j. On the other hand, taking (kN)-th power of the equation (PU) above, we may find elements $\{a_j\} \subset A$ such that

$$a_1 f_1^N + a_2 f_2^N + \cdots + a_k f_k^N = 1$$

holds. Then we compute

$$m - n = (a_1 f_1^N + a_2 f_2^N + \cdots + a_k f_k^N)(m-n) = 0$$

to conclude that $m = n$.

Gluing lemma:

Let $\{m_\lambda \in M_{f_\lambda}\}$ be given such that they satisfy

$$i_{\lambda\mu}(m_\lambda) = i_{\lambda\mu}(m_\mu)$$

for any λ, μ. We fist choose a finite subcovering $\{O_{f_j} = U_{\lambda_j}\}_{j=1}^k$ of $\{U_\lambda\}$. Then we may choose a positive integer N_1 such that

$$m_{\lambda_j} = x_j / f_j^{N_1} \qquad (\exists x_j \in M)$$

holds for all $j \in \{1, 2, 3, \ldots, k\}$.

$$i_{jl}(x_j f_l^{N_1}) = i_{jl}(x_l f_j^N)$$

Then by the same argument which appears in the "locality" part, there exists a positive integer N_2 such that

$$(f_i f_j)^{N_2}(x_j f_l^{N_1} - x_l f_j^{N_1}) = 0$$

holds for all $j, l \in \{1, 2, 3, \ldots, k\}$. We rewrite the above equation as follows.

$$(f_j^{N_2} x_j) f_l^{N_2+N_1} - (f_l^{N_2} x_l) f_j^{N_2+N_1} = 0.$$

On the other hand, by taking $k(N_1 + N_2)$-th power of the equation (PU), we may see that there exist elements $\{b_j\} \in A$ such that

$$\sum_{j=1}^{k} b_j f_j^{N_1+N_2} = 1$$

holds.

Now we put

$$n = \sum_j b_j (f_j^{N_2} x_j).$$

Then since for any l

$$(f_j^{N_2} x_j) = (f_l^{N_2} x_l) f_j^{N_2+N_1} / f_l^{N_2+N_1} = f_j^{N_2+N_1} m_{\lambda_l}$$

holds on O_l, we have $i_l(n) = m_{\lambda_l}$.

Now, take any other open set $O_{f_\mu} = U_\mu$ from the covering $\{U_\lambda\}$. $\{O_{f_j}\}_{j=1}^k \cup \{O_{f_\mu}\}$ is again a finite open covering of $\operatorname{Spec}(A)$. We thus know from the argument above that there exists an element n_1 of M such that

$$i_j(n_1) = m_{f_j}, \quad i_\mu(n_1) = m_\mu.$$

From the locality, n_1 coincides with n. In particular, $i_\mu(n) = m_\mu$ holds. This means n satisfies the requirement for the "glued object".

□

Corollary 1.22. *Let A be a commutative ring. Let B be a non-commutative ring which contains A as a central subalgebra (that means, $Z(B) \supset A$). Then there exists a sheaf $\tilde{B}$ of $\mathcal{O}$-algebras over* $\operatorname{Spec}(A)$

1.6. **Benefit of being a sheaf.** By saying that $\mathcal{O}$ is a sheaf on $\operatorname{Spec}(A)$, we may easily use the arguments we have used to proved the locality and the gluing lemma.

For example, the proof we gave in Theorem 1.16, especially the part where we chose the idempotent p_1, was a bit complicated.

Let us give another proof using the sheaf arguments. There exists a unique element $p \in A = \mathcal{O}(\operatorname{Spec}(A))$ which coincides with 1 on $U_1 = V(J)$ and with 0 on $U_2 = V(I)$. From the uniqueness we see that

$$p^2 = p$$

holds since p^2 satisfies the same properties as p. The rest of the proof is the same.

As a second easier example, we consider the following undergraduate problem.

Problem: Find the inverse of the matrix

$$\begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}.$$

A student may compute (using "operations on rows") as follows.

$$\begin{pmatrix} 3 & 5 & | & 1 & 0 \\ 1 & 2 & | & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 5/3 & | & 1/3 & 0 \\ 1 & 2 & | & 0 & 1 \end{pmatrix}$$
$$\to \begin{pmatrix} 1 & 5/3 & | & 1/3 & 0 \\ 0 & 1/3 & | & -1/3 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 5/3 & | & 1/3 & 0 \\ 0 & 1 & | & -1 & 3 \end{pmatrix}$$
$$\to \begin{pmatrix} 1 & 0 & | & 2 & -5 \\ 0 & 1 & | & -1 & 3 \end{pmatrix}$$

The calculation is valid on Spec($\mathbb{Z}[1/3]$).

Another student may calculate (using "operations on columns") as follows.

$$\begin{pmatrix} 3 & 5 & | & 1 & 0 \\ 1 & 2 & | & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 3 & 5/2 & | & 1 & 0 \\ 1 & 1 & | & 0 & 1/2 \end{pmatrix}$$
$$\to \begin{pmatrix} 1/2 & 5/2 & | & 1 & 0 \\ 0 & 1 & | & -1/2 & 1/2 \end{pmatrix} \to \begin{pmatrix} 1 & 5/2 & | & 2 & 0 \\ 0 & 1 & | & -1 & 1/2 \end{pmatrix}$$
$$\to \begin{pmatrix} 1 & 1 & | & 2 & -5 \\ 0 & 1 & | & -1 & 3 \end{pmatrix}$$

The calculation is valid on Spec($\mathbb{Z}[1/2]$). Of course, both calculations are valid on the intersection $\mathrm{Spec}(\mathbb{Z}[1/2]) \cap \mathrm{Spec}(\mathbb{Z}[1/3]) = \mathrm{Spec}(\mathbb{Z}[1/6])$.

The gluing lemma asserts that the answer obtained individually is automatically an answer on the whole of Spec($\mathbb{Z}$). Of course, in this special case, there are lots of easier ways to tell that. But one may imagine this kind of thing is helpful when we deal with more complicated objects.

1.7. **homomorphisms of (pre)sheaves.**

DEFINITION 1.23. Let $\mathcal{F}_1$, $\mathcal{F}_2$ be presheaves of modules on a topological space X. Then we say that a sheaf homomorphism

$$\varphi : \mathcal{F}_1 \to \mathcal{F}_2$$

is given if we are given a module homomorphism

$$\varphi_U : \mathcal{F}_1(U) \to \mathcal{F}_2(U)$$

for each open set $U \subset X$ with the following property hold.

(1) For any open subsets $V, U \subset X$ such that $V \subset U$, we have

$$\rho_{V,U} \circ \varphi_U = \varphi_V \circ \rho_{V,U}.$$

(The property is also commonly referred to as "φ commutes with restrictions".)

DEFINITION 1.24. A homomorphism of sheaves is defined as a homomorphism of presheaves.

1.8. **example of presheaves and sheafification.** To proceed our theory further, we need to study a bit more about presheaves. Unfortunately, a sheaf of modules $\tilde{M}$ on an affine schemes are "too good". Namely, in terms of cohomology (which we study later,) we have always

$$H^i(\operatorname{Spec}(A), \tilde{M}) = 0 \qquad (\text{if } i > 0).$$

So to study some important problems on sheaf theory (which we will sure to encounter when we deal with non-affine schemes,) we need to study some examples from other mathematical areas.

A first example is a presheaf which satisfies the "locality" of sheaf axiom, but which fails to obey "gluing lemma".

EXAMPLE 1.25. Let $X = \mathbb{R}$ be the (usual) real line with the usual Lebesgue measure. Then we have a presheaf of L^1-functions given by

$$L^1(U) = \{f : U \to \mathbb{C}; |f| \text{ is integrable}\}.$$

L^1 is a presheaf which satisfies the "locality" of sheaf axiom, but which fails to obey "gluing lemma". Indeed, Let $\{U_n = (-n, n)\}$ be an open covering of $\mathbb{R}$ and define a section f_n on U_n by

$$f_n(x) = 1 \qquad (x \in U_n).$$

Then we see immediately that $\{f_n\}$ is a family of sections which satisfies the assumption of "gluing lemma". The function which should appear as the "glued function" is the constant function 1, which fails to be integrable on the whole of $\mathbb{R}$.

We may "sheafificate" the presheaf L^1 above. Instead of L^1-functions we consider functions which are locally L^1. Namely, for any open subset $U \subset \mathbb{R}$, we consider

$$L^1_{\text{loc}}(U) = \left\{ f : U \to \mathbb{C}; \begin{matrix} \forall x \in U, \exists V (\text{open in } U) \ni x \\ \text{such that } |f| \text{ is integrable on } V \end{matrix} \right\}$$

The presheaf so defined is a sheaf, which we may call "the sheaf of locally L^1-functions".

EXAMPLE 1.26. Similarly, we may consider a presheaf $U \mapsto \mathrm{Bdd}(U)$ of bounded functions on a topological space X. We may sheafificate this example and the sheaf so created is the sheaf of locally bounded functions.

EXAMPLE 1.27. It is psychologically a bit difficult to give an example of a presheaf which does not satisfy the locality axiom of a sheaf. But there are in fact a lot of them.

For any differentiable (C^∞) manifold X (students which are not familiar with the manifolds may take X as an open subset of $\mathbb{R}^n$ for an example.), we define a presheaf $\mathcal{G}$ on X defined as follows

$$\mathcal{G}(U) = C^\infty(U \times U) = \{\text{complex valued } C^\infty\text{-functions on } U \times U\}.$$

The restriction is defined in an obvious manner. It is an easy exercise to see that the presheaf does not satisfy the locality axiom of a sheaf.

To sheafificate this, we first need to introduce an equivalence relation on $\mathcal{G}(U)$.

$$f \sim g \iff \begin{pmatrix} \text{there exists an open covering } \{U_\lambda\} \text{ of } U \\ \text{such that } \rho_{U_\lambda,U} f = \rho_{U_\lambda,U} g \\ \text{for any } \lambda. \end{pmatrix}$$

Then we may easily see that

$$f \sim g \iff \begin{pmatrix} \text{there exists an open neighborhood } V \text{ of} \\ \text{the diagonal } \Delta_U \subset U \times U \\ \text{such that } f = g \text{ on } V \end{pmatrix}$$

holds.

Then we define

$$\mathcal{F}(U) = \mathcal{G}(U)/\sim .$$

It is now an easy exercise again to verify that $\mathcal{F}$ so defined is a sheaf. (Readers who are familiar with the theory of jets may notice that the sheaf is related to the sheaf of jets. In other words, there is a sheaf homomorphism from this sheaf to the sheaf of jets.)

1.9. **sheafification of a sheaf.** In the preceding subsection, we have not been explained what "sheafification" really means. Here is the definition.

LEMMA 1.28. *Let $\mathcal{G}$ be a presheaf on a topological space X. Then there exists a sheaf* $\mathrm{sheaf}(\mathcal{G})$ *and a presheaf morphism*

$$\iota_{\mathcal{G}} : \mathcal{G} \to \mathrm{sheaf}(\mathcal{G})$$

such that the following property holds.

(1) *If there is another sheaf* $\mathcal{F}$ *with a presheaf morphism*

$$\alpha : \mathcal{G} \to \mathcal{F},$$

then there exists a unique sheaf homomorphism

$$\tilde{\alpha} : \mathrm{sheaf}(\mathcal{G}) \to \mathcal{F}$$

such that

$$\alpha = \tilde{\alpha} \circ \iota_{\mathcal{G}}$$

holds.

Furthermore, such $\mathrm{sheaf}(\mathcal{G}), \iota_{\mathcal{G}}$ *is unique.*

Definition 1.29. The sheaf $\mathrm{sheaf}(\mathcal{G})$ (together with $\iota_{\mathcal{G}}$) is called the sheafification of $\mathcal{G}$.

The proof of Lemma 1.28 is divided in steps.

The first step is to know the uniqueness of such sheafification. It is most easily done by using universality arguments. ([1] has a short explanation on this topic.)

Then we divide the sheafification process in two steps.

Lemma 1.30. *(First step of sheafification) Let* $\mathcal{G}$ *be a presheaf on a topological space* X*. Then for each open set* $U \subset X$*, we may define a equivalence relation on* $\mathcal{G}(U)$ *by*

$$f \sim g \iff \begin{pmatrix} \textit{there exists an open covering } \{U_\lambda\} \textit{ of } U \\ \textit{such that } \rho_{U_\lambda,U} f = \rho_{U_\lambda,U} g \\ \textit{for any } \lambda. \end{pmatrix}$$

Then we define

$$\mathcal{G}^{(1)}(U) = \mathcal{G}(U)/\sim .$$

Then $\mathcal{G}^{(1)}$ *is a presheaf that satisfies the locality axiom of a sheaf. There is also a presheaf homomorphism from* $\mathcal{G}$ *to* $\mathcal{G}^{(1)}$*. Furthermore,* $\mathcal{G}^{(1)}$ *is universal among such.*

Lemma 1.31. *(Second step of sheafification) Let* $\mathcal{G}$ *be a presheaf on a topological space* X *which satisfies the locality axiom of a sheaf. Then we define a presheaf* $\mathcal{G}^{(2)}$ *in the following manner. First for any open covering* $\{U_\lambda\}$ *of an open set* $U \subset X$*, we define*

$$\mathcal{G}^{(2)}(U; \{U_\lambda\}) = \left\{ \{r_\lambda\} \in \prod_{\lambda \in \Lambda} G(U_\lambda); \begin{array}{l} \rho_{U_{\lambda\mu},U_\mu} f_\mu = \rho_{U_{\lambda\mu},U_\lambda} f_\lambda \\ \textit{for any } \lambda, \mu \in \Lambda. \end{array} \right\}$$

Then we define

$$\mathcal{G}^{(2)}(U) = \varinjlim_{\{U_\lambda\}} \mathcal{G}^{(2)}(U; \{U_\lambda\})$$

Then we may see that $G^{(2)}$ is a sheaf and that there exists a homomorphism from G to $G^{(2)}$. Furthermore, $G^{(2)}$ is universal among such.

Proofs of the above two lemma are routine work and are left to the reader.

Finish of the proof of Lemma 1.28: We put

$$\text{sheaf}(\mathcal{G}) = ((\mathcal{G})^{(1)})^{(2)}$$

□

1.10. **stalk of a presheaf.**

Definition 1.32. Let $\mathcal{G}$ be a presheaf on a topological space X. Let $P \in X$ be a point. We define the stalk of $\mathcal{G}$ on P as

$$\mathcal{G}_P = \varinjlim_{U \ni P} \mathcal{G}(U)$$

It should be noted at this stage that

Lemma 1.33. *Let $\mathcal{G}$ be a presheaf on a topological space X. The natural map*

$$\mathcal{G} \to \text{sheaf}(\mathcal{G})$$

induces an isomorphism of stalk at each point $x \in X$.

1.11. **kernels, cokernels, etc. on sheaves of modules.**

In this subsection we restrict ourselves to deal with sheaves of modules.

To shorten our statements, we call a presheaf which satisfies (only) the sheaf axiom (1) (locality) a "(1)-presheaf".

Lemma 1.34. *Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a homomorphism between sheaves of modules. Then we have*

(1) *The presheaf kernel of φ is a sheaf. We call it the sheaf kernel* $\operatorname{Ker}(\varphi)$ *of φ.*
(2) *The presheaf image of φ is not necessarily a sheaf, but it is a (1)-presheaf. We call the sheafification of the presheaf image as the sheaf image* $\operatorname{Image}(\varphi)$ *of φ.*
(3) *The presheaf cokernel of φ is not necessarily a sheaf. We call the sheafification of the cokernel as the sheaf cokernel* $\operatorname{Coker}(\varphi)$ *of φ.*

Definition 1.35. A sequence of homomorphisms of sheaves of modules

$$\mathcal{F}_1 \xrightarrow{f_1} \mathcal{F}_2 \xrightarrow{f_2} \mathcal{F}_3$$

is said to be exact if $\operatorname{Image}(f_1) = \operatorname{Ker}(f_2)$ holds.

LEMMA 1.36. *A sequence of homomorphisms of sheaves of modules*

$$\mathcal{F}_1 \xrightarrow{f_1} \mathcal{F}_2 \xrightarrow{f_2} \mathcal{F}_3$$

is exact if and only if it is exact stalk wise, that means, if and only if the sequence

$$(\mathcal{F}_1)_P \xrightarrow{f_1} (\mathcal{F}_2)_P \xrightarrow{f_2} (\mathcal{F}_3)_P$$

is exact for all point P.

1.12. **general localization of a commutative ring.** We define a localization of a commutative ring in a more general situation than in subsection 1.3.

DEFINITION 1.37. Let A be a commutative ring. Let S be its subset. We say that S is multiplicative if

(1) $1 \in S$
(2) $x, y \in S \implies xy \in S$

holds.

DEFINITION 1.38. Let S be a multiplicative subset of a commutative ring A. Then we define $A[S^{-1}]$ as

$$A[\{X_s; s \in S\}]/(\{sX_s - 1; s \in S\})$$

where in the above notation X_s is a indeterminate prepared for each element $s \in S$.) We denote by ι_S a canonical map $A \to A[S^{-1}]$.

LEMMA 1.39. *Let S be a multiplicative subset of a commutative ring A. Then the ring $B = A[S^{-1}]$ is characterized by the following property:*

Let C be a ring, $\varphi : A \to C$ be a ring homomorphism such that $\varphi(s)$ is invertible in C for any $s \in S$. Then there exists a unique ring homomorphism $\psi = \phi[S^{-1}] : B \to C$ such that

$$\varphi = \psi \circ \iota_S$$

holds.

COROLLARY 1.40. *Let S be a multiplicative subset of a commutative ring A. Let I be an ideal of A given by*

$$I = \{x \in I; \exists s \in S \text{ such that } sx = 0\}$$

Then (1) I is an ideal of A. Let us put $\bar{A} = A/I$, $\pi : A \to \bar{A}$ the canonical projection. Then:

(2) $\bar{S} = \pi(S)$ is multiplicatively closed.

(3) We have

$$A[S^{-1}] \cong \bar{A}[\bar{S}^{-1}]$$

(4)$\iota_{\bar{S}} : \bar{A} \to \bar{A}[\bar{S}^{-1}]$ is injective.

Example 1.41. $A_f = A[S^{-1}]$ for $S = \{1, f, f^2, f^3, f^4, \dots\}$. The total ring of quotients $Q(A)$ is defined as $A[S^{-1}]$ for

$$S = \{x \in A; x \text{ is not a zero divisor of A}\}.$$

When A is an integral domain, then $Q(A)$ is the field of quotients of A.

Definition 1.42. Let A be a commutative ring. Let $\mathfrak{p}$ be its prime ideal. Then we define the localization of A with respect to $\mathfrak{p}$ by

$$A_{\mathfrak{p}} = A[(A \setminus \mathfrak{p})^{-1}]$$

1.13. **general localization of modules.**

Definition 1.43. Let S be a multiplicative subset of a commutative ring A. Let M be an A-module we may define $S^{-1}M$ as

$$\{(m/s); m \in M, s \in S\}/\sim$$

where the equivalence relation $\sim$ is defined by

$$(m_1/s_1) \sim (m_2/s_2) \iff t(m_1 s_2 - m_2 s_1) = 0 \quad (\exists t \in S).$$

We may introduce a $S^{-1}A$-module structure on $S^{-1}M$ in an obvious manner.

$S^{-1}M$ thus constructed satisfies an universality condition which the reader may easily guess.

1.14. **local rings.**

Definition 1.44. A commutative ring A is said to be a local ring if it has only one maximal ideal.

Example 1.45. We give examples of local rings here.

- Any field is a local ring.
- For any commutative ring A and for any prime ideal $\mathfrak{p} \in \operatorname{Spec}(A)$, the localization $A_{\mathfrak{p}}$ is a local ring with the maximal ideal $\mathfrak{p}A_{\mathfrak{p}}$.

Lemma 1.46. (1) *Let A be a local ring. Then the maximal ideal of A coincides with $A \setminus A^\times$.*
(2) *A commutative ring A is a local ring if and only if the set $A \setminus A^\times$ of non-units of A forms an ideal of A.*

Proof. (1) Assume A is a local ring with the maximal ideal $\mathfrak{m}$. Then for any element $f \in A \setminus A^\times$, an ideal $I = fA + \mathfrak{m}$ is an ideal of A. By Zorn's lemma, we know that I is contained in a maximal ideal of A. From the assumption, the maximal ideal should be $\mathfrak{m}$. Therefore, we have

$$fA \subset \mathfrak{m}$$

which shows that

$$A \setminus A^\times \subset \mathfrak{m}.$$

The converse inclusion being obvious (why?), we have

$$A \setminus A^\times = \mathfrak{m}.$$

(2) The "only if" part is an easy corollary of (1). The "if" part is also easy.

□

Corollary 1.47. *Let A be a commutative ring. Let $\mathfrak{p}$ its prime ideal. Then $A_\mathfrak{p}$ is a local ring with the only maximal ideal $\mathfrak{p}A_\mathfrak{p}$.*

Proposition 1.48. *Let A be a commutative ring. Let $\mathfrak{p} \in \operatorname{Spec}(A)$ then the stalk $\mathcal{O}_\mathfrak{p}$ of $\mathcal{O}$ on $\mathfrak{p}$ is isomorphic to $A_\mathfrak{p}$.*

Definition 1.49. Let A, B be local rings with maximal ideals $\mathfrak{m}_A, \mathfrak{m}_B$ respectively. A local homomorphism $\varphi : A \to B$ is a homomorphism which preserves maximal ideals. That means, a homomorphism φ is said to be local if

$$\varphi^{-1}(\mathfrak{m}_B) = \mathfrak{m}_A$$

Example 1.50 (of NOT being a local homomorphism).

$$\mathbb{Z}_{(p)} \to \mathbb{Q}$$

is not a local homomorphism.

2. inverse images of sheaves

2.1. inverse image of a sheaf.

Definition 2.1. Let $f : X \to Y$ be a continuous map between topological spaces. Let $\mathcal{F}$ be a sheaf on Y. Then the inverse image $f^{-1}\mathcal{F}$ of $\mathcal{F}$ by f is the sheafification of a presheaf $\mathcal{G}$ defined by

$$\mathcal{G}(U) = \varinjlim_{V \supset f(U)} \mathcal{F}(V).$$

Lemma 2.2. *Let $f : X \to Y$ be a continuous map between topological spaces. Let $\mathcal{F}$ be a sheaf on Y. Then we have a natural isomorphism*

$$f^{-1}(\mathcal{F})_x \cong \mathcal{F}_{f(x)}$$

for each point $x \in X$.

Proof. Let $\mathcal{G}$ be the presheaf defined as in the previous Definition. Since sheafification does not affect stalks, we have a natural isomorphism

$$f^{-1}(\mathcal{F})_x \cong \mathcal{G}_x$$

On the other hand, we have

$$\mathcal{G}_x = \varinjlim_{U \ni x} \mathcal{G}(W) = \varinjlim_{U \ni x} \left(\varinjlim_{V \supset f(U)} \mathcal{F}(V) \right)$$

Then since the map f is continuous, the injective limit at the right hand side may be replaced by

$$\varinjlim_{V \ni f(x)} \mathcal{F}(V) = \mathcal{F}_{f(x)}$$

□

DEFINITION 2.3. A ringed space $(X, \mathcal{O}_X)$ is a topological space X with a sheaf of rings $\mathcal{O}_X$ on it. A locally ringed space is a ringed space whose stalks are local rings.

DEFINITION 2.4. Let $(X, \mathcal{O}_X)$ $(Y, \mathcal{O}_Y)$ be ringed spaces.

(1) A morphism $(f, f^\#) : X \to Y$ as ringed spaces is a continuous map $f : X \to Y$ together with a sheaf homomorphism

$$f^\# : f^{-1}(\mathcal{O}_Y) \to \mathcal{O}_X.$$

(Note that $f^\#$ gives a ring homomorphism

$$f_x^\# : O_{Y,f(x)} \to O_{X,x}$$

for each point $x \in X$. We call it an "associated homomorphism".)

(2) Let us further assume that X, Y are locally ringed space. Then a morphism $(f, f^\#)$ of ringed spaces is said to be a morphism of locally ringed spaces if the associated homomorphism $f_x^\#$ is a local homomorphism for each point $x \in X$.

It goes without saying that when X is a (locally) ringed space, then its open set U also carries a structure of (locally) ringed space in a natural way, and that the inclusion map $U \to X$ is a morphism of (locally) ringed space.

2.2. **Definition of schemes.**

DEFINITION 2.5. A scheme is a locally ringed space which is locally isomorphic (as a locally ringed space) to a spectrum $\mathrm{Spec}(A_U)$ of a ring A_U.

3. TENSOR PRODUCTS AND INVERSE IMAGE OF SHEAVES

3.1. tensor products of modules over an algebra.

DEFINITION 3.1. Let A be a (not necessarily commutative) ring. Let M be a right A-module. Let N be a left A-module. Then we define the tensor product of M and N over A, denoted by

$$M \otimes_A N$$

as a module generated by symbols

$$\{m \otimes n; m \in M, n \in N\}$$

with the following relations.

(1)
$$(m_1 + m_2) \otimes n = m_1 \otimes n + m_2 \otimes n \quad (m_1, m_2 \in M,\ n \in N)$$
(2)
$$m \otimes (n_1 + n_2) = m \otimes n_1 + m \otimes n_2 \quad (m \in M,\ n_1, n_2 \in N)$$
(3)
$$ma \otimes n = m \otimes an \qquad (m \in M,\ n \in N,\ a \in A)$$

3.2. universality of tensor products.

DEFINITION 3.2. Let A be a (not necessarily commutative) ring. Let M be a right A-module. Let N be a left A-module. Then for any module X, a map $f : M \times N \to X$ is said to be an A-balanced biadditive map if it satisfies the following conditions.

(1) $f(m_1 + m_2, n) = f(m_1, n) + f(m_2, n) \quad (\forall m_1, m_2 \in M, \forall n \in N)$
(2) $f(m, n_1 + n_2) = f(m, n_1) + f(m, n_2) \quad (\forall m \in M, \forall n_1, n_2 \in N)$
(3) $f(ma, n) = f(m, an) \quad (\forall m \in M, \forall n \in N, \forall a \in A)$

LEMMA 3.3. *Let A be a (not necessarily commutative) ring. Let M be a right A-module. Let N be a left A-module. Then for any module X, there is a bijective additive correspondence between the following two objects.*

(1) *An A-balanced bilinear map $M \times N \to X$*
(2) *An additive map $M \otimes_A N \to X$*

3.3. additional structures on tensor products.

LEMMA 3.4. *Let A be a (not necessarily commutative) ring. Let M be a right A-module. Let N be a left A-module. If M carries a structure of an A-algebra, then the tensor product $M \times_A N$ carries a structure of M-module in the following manner.*

$$x.(y \otimes n) = (xy) \otimes n \qquad (x, y \in M, n \in N)$$

3.4. **tensor products and localizations.**

LEMMA 3.5. *Let A be a commutative ring. Let M be an A-module. Then we have a canonical isomorphism of A_S module*

$$A_S \otimes_A M \cong M_S.$$

3.5. **tensor products of sheaves of modules.**

DEFINITION 3.6. Let $(X, \mathcal{A})$ be a sheaf of algebras (possibly non commutative). Let $\mathcal{F}$ be a right $\mathcal{A}$-module. Let $\mathcal{G}$ be a left $\mathcal{A}$-module. Then the tensor product $\mathcal{F} \otimes_{\mathcal{A}} \mathcal{G}$ is the sheafification of the presheaf defined by

$$U \mapsto \mathcal{F}(U) \otimes_{\mathcal{A}(U)} \mathcal{G}(U).$$

DEFINITION 3.7. Let $f : X \to Y$ be a morphism between locally ringed spaces. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_Y$-modules on Y. Then the **inverse image of $\mathcal{F}$ as an $\mathcal{O}$-module** with respect to f as a sheaf of $\mathcal{O}$-modules is defined as

$$f^*(\mathcal{F}) = f^{-1}(\mathcal{F}) \otimes_{f^{-1}(\mathcal{O}_Y)} \mathcal{O}_X$$

3.6. **sheaves associated to modules.** Let A be a ring. Then we have learned that for each A-module M, there exists a module $\tilde{M}$ associated to it. By using tensor product, we may express this sheaf as follows.

$$\tilde{M} \cong \mathcal{O}_X \otimes_A M$$

To avoid confusion, we prefer the expression $\mathcal{O}_X \otimes_A M$ for the later discussions.

3.7. **direct image of a sheaf.**

DEFINITION 3.8. Let X, Y be topological spaces. Let $f : X \to Y$ be a continuous map. Let $\mathcal{F}$ be a sheaf on X. Then we define its direct image with respect to f by

$$f_*(\mathcal{F})(U) = \mathcal{F}(f^{-1}(U))$$

with obvious restriction maps.

PROPOSITION 3.9. *Let X, Y be topological spaces. Let $f : X \to Y$ be a continuous map. Let $\mathcal{F}$ be a sheaf on X. Let $\mathcal{G}$ be a sheaf on Y. Then we have a natural isomorphism.*

$$\mathrm{Hom}(\mathcal{G}, f_*\mathcal{F}) \cong \mathrm{Hom}(f^{-1}\mathcal{G}, \mathcal{F})$$

PROOF. We first define an adjoint map

$$\iota : f^{-1} f_* \mathcal{F} \to \mathcal{F}$$

and construct the isomorphism using it.

□

PROPOSITION 3.10. *Let X, Y be (locally) ringed spaces. Let $f : X \to Y$ be a morphism of (locally) ringed spaces. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Let $\mathcal{G}$ be a sheaf on $\mathcal{O}_Y$-modules. Then we have a natural isomorphism of modules.*

$$\mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{G}, f_*\mathcal{F}) \cong \mathrm{Hom}_{\mathcal{O}_X}(f^*\mathcal{G}, \mathcal{F})$$

PROOF.

$$\begin{aligned}&\mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{G}, f_*\mathcal{F}) \cong \mathrm{Hom}_{f^{-1}\mathcal{O}_Y}(f^{-1}\mathcal{G}, \mathcal{F})\\ &\cong \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{G}, \mathcal{F}) \cong \mathrm{Hom}_{\mathcal{O}_X}(f^*\mathcal{G}, \mathcal{F})\end{aligned}$$

□

EXAMPLE 3.11. Let A, B be rings. Let $\varphi : A \to B$ be a ring homomorphism. We put $f = \mathrm{Spec}(\varphi)$ be the continuous map $Y = \mathrm{Spec}(B) \to \mathrm{Spec}(A) = X$ corresponding to φ. We note that B carries an A-module structure via φ. Accordingly, we have the corresponding sheaf $\mathcal{O}_X \otimes_A B$ on X. We may easily see that this sheaf coincides with $f_*\mathcal{O}_Y$. The map $\varphi : A \to B$ then may also be regarded as a homomorphism of A-modules. We have thus an $\mathcal{O}_X$ module homomorphism

$$\varphi_\# : \mathcal{O}_X \to f_*\mathcal{O}_Y$$

of sheaves on X. By the adjoint relation (Proposition 3.9), we obtain a sheaf homomorphism

$$\mathrm{Spec}(\varphi)^\# : f^\#\mathcal{O}_X \to \mathcal{O}_Y.$$

of sheaves of rings.

4. AFFINE SCHEMES AND RINGS: EQUIVALENCE OF CATEGORIES

DEFINITION 4.1. Let A, B be rings. Let $\varphi : A \to B$ be a ring homomorphism. We have already introduced $\mathrm{Spec}(\varphi)$ as a continuous map $Y = \mathrm{Spec}(B) \to \mathrm{Spec}(A) = X$. Now that the spaces $\mathrm{Spec}(A), \mathrm{Spec}(B)$ carry structures of locally ringed spaces, we (re)define $\mathrm{Spec}(\varphi)$ as a morphism of locally ringed spaces by defining $\mathrm{Spec}(\varphi)^\#$ as in Example 3.11.

LEMMA 4.2. $\mathrm{Spec}(\varphi)$ *is indeed a morphism of locally ringed space.*

THEOREM 4.3. *Let $(f, f^\#) : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ be a morphism of locally ringed space. Then there exists an unique ring homomorphism $\varphi : A \to B$ such that f coincides with* $\mathrm{Spec}(\varphi)$.

PROOF. Let us put $Y = \mathrm{Spec}(B)$ and $X = \mathrm{Spec}(A)$. The data

$$f^\# : f^{-1}\mathcal{O}_X \to \mathcal{O}_Y$$

is equivalent to a data

$$f_{\#} : \mathcal{O}_X \to f_* \mathcal{O}_Y$$

which gives rise to a ring homomorphism

$$f_{\#}(X) : A = \mathcal{O}_X(X) \to (f_* \mathcal{O}_Y)(X) = B.$$

Let us take this homomorphism as φ.

$$\begin{array}{ccc} A_{f(y)} = \mathcal{O}_{X,f(y)} & \xrightarrow{f_y^{\#}} & \mathcal{O}_{Y,y} = B_y \\ {\scriptstyle \mathrm{restr}_{f(y)}}\uparrow & & {\scriptstyle \mathrm{restr}_y}\uparrow \\ \mathcal{O}_X(X) & \xrightarrow{\varphi} & \mathcal{O}_Y(Y) \end{array}$$

By the hypothesis of f being a morphism of locally ringed spaces, $f_y^{\#}$ is local homomorphism. That means,

$$(f^{\#})^{-1}(yB_y) = f(y)A_{f(y)}.$$

$$\varphi^{-1}(y) = f(y).$$

From the definition of Spec(φ), we have

$$\mathrm{Spec}(\varphi)(y) = f(y).$$

We have thus proved that Spec(φ) is equal to f as a map $Y \to X$. □

References

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[2] W. C. Waterhouse, *Introduction to affine group schemes*, Springer Verlag, 1997.